12.333 Problem Set 2: Tides and vorticity (due Thursday March 19)

1. a) Planet A, a perfect sphere of uniform density, of mass $M_A$ and radius $a_A$, is subjected to tidal forces induced by a star B, of mass $M_B$ and radius $a_B$, around which A orbits at a distance $R$. Given the expression for $Z_e$ we derived in class, show that the “equilibrium tide” $Z_e$ (in the sense we defined in class) on planet A is given by

$$Z_e = \frac{3}{2} \frac{M_B}{M_A} \left( \frac{a_A}{R} \right)^3.$$ 

b) Consider the Earth as a perfect sphere of radius $a = 6371$km, covered by an ocean of mean depth 4km. Assume the moon does not exist; the only tide is due to the Sun. Assume (wrongly, but assume it anyway) that the solar day is long enough for the ocean tide to be given by the “equilibrium tide.” How close to the Sun would the Earth have to be for the ocean depth at the equator to vanish at low tide? [Mass of Sun $= 1.99 \times 10^{30}$kg; mass of Earth $= 5.97 \times 10^{24}$kg]

2. Using the attached figure, estimate the e-folding distance from the coast of tidal amplitude, and the phase speed along the E coast of Great Britain. Discuss the extent to which these quantities (the e-folding distance and phase speed) are mutually consistent and use them to estimate the typical water depth. [For a length scale, note that 1deg latitude $\approx 111$km.]

3. Consider a sphere (such as the Earth) rotating with angular velocity $\Omega$. Relative to the rotating sphere, the atmosphere is stationary. Express the velocity components in the atmosphere relative to an inertial reference frame (but still in Earth-centered spherical coordinates). Given that, in spherical geometry, the vertical component of vorticity is

$$\zeta = \frac{1}{a \cos \varphi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \varphi} (u \cos \varphi) \right),$$

where $\lambda$ is longitude, $\varphi$ latitude, $a$ is the Earth’s radius, and $(u, v)$ the eastward and northward velocity components, show that the vertical component of vorticity of this atmosphere at each latitude, expressed relative to an inertial reference frame, is equal to the local Coriolis parameter.

4. Consider the “point vortex” flow we discussed in class (Section 4.3.3). Show that the angular momentum (per unit mass) of this flow is independent of $r$. 