6.2 The continuous case: vertical structure of equatorial waves

Consider 3D adiabatic, inviscid motions on an equatorial $\beta$-plane. The equations of motion are

\[
\frac{du}{dt} - \beta y v = - \frac{\partial \phi}{\partial x} \tag{14}
\]

\[
\frac{dv}{dt} + \beta y u = - \frac{\partial \phi}{\partial y}.
\]

Take a basic state of no motion ($\bar{u} = \bar{v} = \bar{w} = 0$); then linearize to

\[
\frac{\partial u'}{\partial t} - \beta y v' = - \frac{\partial \phi'}{\partial x} \tag{15}
\]

\[
\frac{\partial v'}{\partial t} + \beta y u' = - \frac{\partial \phi'}{\partial y}.
\]

Similarly, our continuity, thermodynamic and hydrostatic log-pressure equations linearize to give

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w') = 0, \tag{16}
\]

\[
\frac{\partial T'}{\partial t} + w' \frac{H}{R} N^2 = 0 \tag{17}
\]

\[
\frac{\partial \phi'}{\partial z} = \frac{R}{H} \frac{T'}{T}, \tag{18}
\]

where, as before, the buoyancy frequency $N$ is defined by

\[
N^2 = \frac{R}{H} \left( \frac{dT}{dz} + \frac{\kappa}{H} \frac{T}{T} \right)
\]

which we shall assume to be a function of $z$ only. We can use (17) and (18) to express $w'$ in terms of $\phi'$:

\[
w' = - \frac{R}{N^2 H} \frac{\partial T'}{\partial t} = - \frac{1}{N^2} \frac{\partial^2 \phi'}{\partial z \partial t} \tag{19}
\]
and thereby eliminate \( w' \) from (16). Then we proceed in our usual way. The coefficients are functions of \( y \) and \( z \), so we look for solutions

\[
\begin{bmatrix}
  u' \\
v' \\
w' \\
\phi' \\
T'
\end{bmatrix} = \text{Re} \begin{bmatrix}
  U(y, z) \\
  V(y, z) \\
  W(y, z) \\
  \Phi(y, z) \\
  \Theta(y, z)
\end{bmatrix} e^{i(kx-\omega t)}.
\]

We use (15), (16) and (19) to give

\[
\begin{align*}
-i\omega U - \beta y V &= -ik \Phi; \\
-i\omega V + \beta y U &= -\frac{\partial \Phi}{\partial y}; \\
ikiU + \frac{\partial V}{\partial y} + \frac{i\omega}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{\partial \Phi}{N^2 \partial z} \right) &= 0.
\end{align*}
\]

(20)

Now we go further and look for separable solutions such that

\[
\begin{align*}
U(y, z) &= \tilde{U}(y) Z(z); \\
V(y, z) &= \tilde{V}(y) Z(z); \\
\Phi(y, z) &= \tilde{\Phi}(y) Z(z).
\end{align*}
\]

Then the vertical structure function \( Z(z) \) must satisfy

\[
\frac{d}{dz} \left[ \frac{\rho}{N^2} \frac{dZ}{dz} \right] = -\rho \alpha^2 Z,
\]

(21)

where \( \alpha^2 \) is the separation constant.

As usual, in an unbounded atmosphere, (21) has vertically propagating solutions of the form \( Z(z) \simeq \rho^{-1/2} e^{imz} \); such waves do exist and, in fact, are very important in the tropical stratosphere where the associated vertical transport of momentum plays a major role in the momentum budget. For such waves, vertical wavenumber \( m \) is related to the separation constant by

\[
m = \pm \left( \alpha^2 N^2 - \frac{1}{4H^2} \right)^{1/2},
\]

the two signs corresponding to upward and downward propagating components. If, however, we consider the troposphere as a bounded system, with
a lid at the tropopause\textsuperscript{1} ($z = D$, say), then we get vertical *modes* and the vertical structure eq. (21) becomes an eigenvalue problem.

As we discussed in the context of 3D, QG Rossby waves, the boundary condition in our log-p coordinate system on a rigid horizontal boundary at $z = z_B$ is not $w' = 0$ but

$$w' = -\frac{1}{g} \left( \frac{\partial \phi'}{\partial t} \right) \text{ on } z = z_B,$$

which leads us to

$$\frac{\partial}{\partial t} \left( \frac{\partial \phi'}{\partial z} - \frac{N^2}{g} \phi' \right) = 0 \text{ on } z = z_B.$$

With our formulation $\phi' = Re \tilde{\Phi}(y)Z(z)e^{i(kx-\omega t)}$ and assuming $\omega$ to be nonzero:

$$\frac{dZ}{dz} - \frac{N^2}{g} Z = 0 \text{ on } z = z_B. \quad (22)$$

Just as we found for the Rossby wave problem, solutions to (21) in an atmosphere, unbounded as $z \to \infty$ and with constant $N^2$, are of two types. The first solution (which we designate $n = 0$) is the external mode, described by

$$Z_0(z) = \text{const } e^{N^2z/g}, \quad (23)$$

which is a solution provided

$$\alpha_0^2 = \frac{1}{gH} \left( 1 - \frac{N^2H}{g} \right). \quad (24)$$

In general, this mode is insensitive to the existence of an upper boundary (if one existed)

Suppose now the atmosphere has a lid at $z = D$. Then the propagating continuum solutions are no longer appropriate; we now get an infinite set of discrete modes which satisfy the boundary condition (22) at $z_B = 0$ and $z_B = D$. The external mode $n = 0$ is unaffected by the upper boundary, but we now have the *internal*, or *baroclinic*, modes $n \geq 1$ for which

$$Z_n = \text{const } e^{z^2/2H} \left( \sin \frac{n\pi z}{D} - \frac{n\pi g}{N^2D} \cos \frac{n\pi z}{D} \right). \quad (25)$$

\textsuperscript{1}Of course, there is no such lid, nor does the stratosphere act as a lid. We do this for reasons to become apparent later.
The corresponding separation constants $\alpha_n$ are defined by

$$\alpha_n^2 = \frac{1}{N^2} \left( \frac{n^2 \pi^2}{D^2} + \frac{1}{4H^2} \right). \quad (26)$$

Thus we have an infinite set of vertical modes, $n = 0$ corresponding to the external mode and the rest being the $n^{th}$ baroclinic modes. Note that the existence of these baroclinic modes relies crucially on the existence of the upper boundary. Remember, however, that even when allowance is made for nonuniform stratification, in reality, under almost all realistic circumstances, the unbounded atmosphere possesses only the external mode.

Forms of $Z_n(z)$ are as shown:

As for the Rossby wave case, the horizontal structure problem we are left with, from (20) with (21) is

$$-i\omega U - \beta y V = -ik\Phi;$$

$$-i\omega V + \beta y U = -\frac{\partial\Phi}{\partial y};$$

$$ikU + \frac{\partial V}{\partial y} - i\omega \alpha^2 \Phi = 0. \quad (27)$$

This is precisely the same set of equations we obtained for the shallow water system (with $gh_0 \to \Phi$), of depth $h_e$, where

$$gh_e = \alpha^2. \quad (28)$$

So each eigenvalue $\alpha_n$ can equivalently be written as

$$\alpha_n^2 = (gh_n)^{-1} = c_n^{-2},$$

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where $h_n$ is the equivalent depth of the $n^{th}$ mode. Recall also that the equatorial deformation radius, $L_E = (gh_n)^{1/4} \beta^{-1/2} = \sqrt{c_n/\beta}$ is also different for each mode. For our hypothetical bounded system, and taking $D = 15 km$, $H = 7.5 km$, $N^2 = 1.5 \times 10^{-4} s^{-2}$, these indices are, for the first few modes:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h_n$ (km)</th>
<th>$c_n$ (ms$^{-1}$)</th>
<th>$L_{E,n}$ (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.471</td>
<td>288.3</td>
<td>3620</td>
</tr>
<tr>
<td>1</td>
<td>0.316</td>
<td>55.7</td>
<td>1590</td>
</tr>
<tr>
<td>2</td>
<td>0.085</td>
<td>28.8</td>
<td>1140</td>
</tr>
</tbody>
</table>

So, for each vertical mode, there is an infinite set of latitudinal modes: for each $n$ (and for the continuum solutions) we have a Kelvin wave, the mixed Rossby-gravity wave, and the full set of Rossby and gravity waves.