Instability of zonal flows (QG)

(Holton, Ch. 8; a detailed exposition of the theory is given in Pedlosky, Ch 7.7.)

Violating the stability constraint: Barotropic and baroclinic instability

As we have seen zonal flows are stable to, inviscid, adiabatic, normal mode, QG disturbances if the PV gradient is single signed and the upper and lower boundaries are isentropic. Since the QGPV is

\[ q = f + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \]

and for a zonal flow \( U = -\partial \psi / \partial y \), the PV gradient is

\[ \frac{\partial q}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \right) \] (1)

For a barotropic flow (no \( T \) gradients, and \( \partial U / \partial z = 0 \)), non-isentropic boundaries are not an issue, and the PV gradient is

\[ \left( \frac{\partial q}{\partial y} \right)_{\text{barotropic}} = \beta - \frac{\partial^2 U}{\partial y^2} \] (2)

In the case \( \beta = 0 \), such a zonal flow is necessarily stable unless the curvature term has both signs within the fluid. If the curvature is everywhere finite, this yields the inflection point theorem: the flow is stable unless \( U(y) \) has an inflection point where \( \partial^2 U / \partial y^2 = 0 \). Introduction of \( \beta > 0 \) is a stabilizing influence: barotropic instability is then possible only if the curvature term is of the correct sign and of sufficient magnitude to overcome \( \beta \) in (2), somewhere in the flow.

In the absence of barotropic curvature,

\[ \frac{\partial q}{\partial y} = \beta - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \right) \] (3)

and baroclinic instability is possible if the upper and lower boundaries are not isentropic, or if the vertical “curvature” (modified by \( \rho f_0^2 / N^2 \)) is sufficient
to overcome $\beta$ in (3). In the extratropical troposphere, the curvature is not usually sufficient to do so in the free atmosphere, but the stability condition is violated by the presence of temperature gradients on the lower boundary.

In the presence of both barotropic and baroclinic curvature, both curvature terms in (1) may contribute to changing the sign of $\partial q/\partial y$, resulting in mixed barotropic-baroclinic instability.

**Baroclinic instability: The Eady problem**

The simplest example of baroclinic instability (and one which is actually more relevant to the real atmosphere than it might appear) in a continuously stratified fluid is the *Eady* problem. The mean state has the following characteristics:

1. It is Boussinesq ($\rho = \text{constant in inertial terms}$)
2. Inviscid, adiabatic flow on an $f-$ plane ($f = f_0$ is constant: $\beta = 0$)
3. Uniform buoyancy frequency: $N^2 = -g\rho_0^{-1}(\partial \rho_0/\partial z)$ is constant
4. Rigid horizontal upper and lower boundaries at $z = \pm \frac{1}{2}D$, on which $w = 0$.
5. Basic state comprises a zonal flow that increases linearly with height: $u_0 = \Lambda z$
6. Basic state density in thermal wind balance with the wind:

$$\rho_0 = \rho_{00} \left[ 1 + \frac{1}{g} \left( f \Lambda y - N^2 z \right) \right]$$

where $\rho_{00}$ is constant, so $\partial \rho_0/\partial y = f \Lambda \rho_{00}/g$: $\rho_0$ increases uniformly with latitude everywhere.

The basic state geostrophic streamfunction is

$$\Psi = -\int U(z) \, dy = -\Lambda z y + \text{constant}$$

and the basic state QGPV is

$$Q = f_0 + \frac{\partial^2 \Psi}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \Psi}{\partial z^2} = f_0$$
so the basic state has no PV gradient—this is the defining characteristic of the Eady problem. It then follows from the stability criterion that this flow must be stable unless there are density gradients on the boundaries, which of course there are.

The perturbation QGPV equation is

\[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) q' + v' \frac{\partial Q}{\partial y} = 0. \]

But since \( \frac{\partial Q}{\partial y} = 0 \), if \( q' = 0 \) everywhere at some initial time, then

\[ q' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \psi'}{\partial z^2} = 0. \]

If we look for separable modal solutions, wave-like in the horizontal, of the form

\[ \psi' = \text{Re} \left[ \Phi(z) e^{i(kx + ly - kt)} \right] \]

then

\[ \frac{d^2 \Phi}{dz^2} - \frac{N^2}{f_0^2} \kappa^2 \Phi = 0 \]

where \( \kappa = \sqrt{k^2 + l^2} \). Then \( \Phi \sim \exp(\pm N \kappa z / f_0) \), or

\[ \Phi(z) = A \cosh \left( \frac{N \kappa}{f_0} z \right) + B \sinh \left( \frac{N \kappa}{f_0} z \right). \]

Thus, we can regard this as the sum of two exponentials, decaying away from the lower and upper boundaries.

To close the problem, we need to invoke the upper and lower boundary conditions \( w' = 0 \). To do so, consider the thermodynamic equation, which for this Boussinesq system is just \( d\rho/dt = 0 \), which yields the linearized perturbation equation

\[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \rho' + v' \frac{\partial \rho_0}{\partial y} + w' \frac{\partial \rho_0}{\partial z} = 0. \]

Since \( \rho' = -(f_0 \rho_{00}/g) \partial \psi'/\partial z \), and \( \partial \rho_0/\partial y = f \Lambda \rho_{00}/g \), and \( w' = 0 \) on the boundaries, we have

\[ (U - c) \frac{d\Phi}{dz} - \Lambda \Phi = 0 \]

(6)
on each boundary.

It is convenient here to define the length scale, \( L = ND/f_0 \), the *internal radius of deformation*. This compares with the external radius deformation, \( L_e = \sqrt{gD/f_0} \):
\[
L = \frac{ND}{f_0} = \frac{D}{f_0} \sqrt{\frac{g}{\rho_{00}}} \left| \frac{d\rho_0}{dz} \right| = \frac{D}{f_0} \sqrt{\frac{g}{D} \frac{\Delta\rho_0}{\rho_{00}}} = L_e \sqrt{\frac{\Delta\rho_0}{\rho_{00}}}.
\]

Then \( N\kappa z/f_0 = \kappa L z/D \).

Now, applying (5) to each boundary condition (6) in turn, noting that \( U = \Lambda D/2, -\Lambda D/2 \), on the upper and lower boundaries respectively,
\[
\kappa \frac{L}{D} \left( \frac{\Lambda D}{2} - c \right) \left[ A \sinh \left( \frac{1}{2} \kappa L \right) + B \cosh \left( \frac{1}{2} \kappa L \right) \right] \\
-\Lambda \left[ A \cosh \left( \frac{1}{2} \kappa L \right) + B \sinh \left( \frac{1}{2} \kappa L \right) \right] = 0
\]
\[
-\kappa \frac{L}{D} \left( \frac{\Lambda D}{2} + c \right) \left[ -A \sinh \left( \frac{1}{2} \kappa L \right) + B \cosh \left( \frac{1}{2} \kappa L \right) \right] \\
-\Lambda \left[ A \cosh \left( \frac{1}{2} \kappa L \right) - B \sinh \left( \frac{1}{2} \kappa L \right) \right] = 0
\]  

Eq. (7) represents an eigenvalue problem for \( c \). After a good deal of manipulation (see the Appendix for details) we find
\[
c = \pm \frac{\Lambda D}{\kappa L} \sqrt{\left[ \frac{\kappa L}{2} - \tanh \left( \frac{1}{2} \kappa L \right) \right] \left[ \frac{\kappa L}{2} - \coth \left( \frac{1}{2} \kappa L \right) \right]}.
\]  

The function \((x - \tanh x)(x - \coth x)\) is plotted in Fig. 1. When \( x < 1.1997 \), the function is negative\(^1\), and \( c \) is then purely imaginary; when \( x > 1.1997 \), \( c \) is purely real. Note that since our solution (4) depends on time as \( \exp(-ikct) \), we have propagating waves, without growth or decay, for

\(^1\)In fact, the zero of the function occurs where \( x = \coth x \).
Figure 1: The function $y = (x - \tanh x)(x - \coth x)$.

$\text{Im} \,(c) = 0$, and growing\footnote{Since (8) has two solutions of opposite signs, whenever there is a growing solution, there is also a corresponding decaying solution. But the growing solution with positive Im part is the interesting one.} waves for $\text{Im} \,(c) > 0$. Since, from (8), we have

$$\frac{c}{\Lambda D} = \pm \frac{1}{\kappa L} \sqrt{\left[ \frac{\kappa L}{2} - \tanh \left( \frac{1}{2} \kappa L \right) \right] \left[ \frac{\kappa L}{2} - \coth \left( \frac{1}{2} \kappa L \right) \right]} \quad (9)$$

we can plot $c/\Lambda D$ vs. $\kappa L$; this is shown in Fig. 2.

The gross characteristics of the solutions, therefore, depend solely on whether or not $\mu D = f_0 \kappa D / N$ exceeds the value $\gamma_0 = 2.3994$. For given $N$ and $D$, the long waves grow:

$$\kappa < \gamma_0 L^{-1}, \quad \text{Im} \,(c) \neq 0$$

$$\kappa > \gamma_0 L^{-1}, \quad \text{Im} \,(c) = 0$$

For very short waves, $\kappa L \gg 1$ and (8) gives us

$$c \to \pm \frac{1}{2} \Lambda D.$$
These “Eady edge waves” (which are formally equivalent to Rossby waves, owing their existence to the temperature gradients there) are trapped at each boundary, and each is simply advected by the local flow. For smaller $\kappa$, the two boundary waves decay less rapidly with height and interact, slowing each other’s propagation. Eventually, when $\kappa$ exceeds the critical value, this interaction stalls the waves ($\text{Re} \,(c) \to 0$, so the waves propagate at the speed of the mid-level flow, which happens to be zero in this case) and they begin to reinforce each other, causing growth of the coupled boundary waves.

Even though $c$ depends on wavenumber only through its magnitude $\kappa$, the growth rate of growing waves, $\sigma = k \text{Im} \,(c)$, is a function of $k$ and $l$. Writing dimensionless wavenumbers $k' = kL$, $l' = lL$, $\kappa' = \sqrt{k'^2 + l'^2} = \kappa L$, we have

$$\frac{\sigma L}{\Lambda D} = k' \text{Im} \left( \frac{c}{\Lambda D} \right)$$

and so from (9) we have

$$\frac{\sigma N}{f_0 \Lambda} = \pm \left( \frac{k'}{\kappa'} \right) \sqrt{\left[ \frac{\kappa'}{2} - \tanh \left( \frac{1}{2} \kappa' \right) \right] \left[ \frac{\kappa'}{2} - \coth \left( \frac{1}{2} \kappa' \right) \right]}.$$
The dependence of $\sigma N/f_0\Lambda$ on $k'$ and $l'$ for the growing wave is plotted in Fig. 3. The maximum growth rate, $\sigma N/f_0\Lambda = 0.31$, is found at $k = 1.61L^{-1}$, $l = 0$. Note that the growth rate $\sigma$ depends on the ratio $\Lambda/f_0N$, and therefore, as one might expect, increases with increasing baroclinic shear $\Lambda$, but that the wavelength of the fastest growing wave is independent of $\Lambda$. Is this instability relevant to the real world? In the midlatitude troposphere, $D \approx 10\text{km}$, $N \approx 1.2 \times 10^{-2}\text{s}^{-1}$, $f_0 \approx 1.0 \times 10^{-4}\text{s}^{-1}$, and $\Lambda$ is typically $25\text{ms}^{-1}/10\text{km} \approx 2.5 \times 10^{-3}\text{s}^{-1}$. So the fastest growth rate is $0.31 \times 2.5 \times 10^{-7}/1.2 \times 10^{-2} \approx 6.5 \times 10^{-6}\text{s}^{-1}$, or an e-folding time scale of $1.5 \times 10^5\text{s} \approx 1.8$ days. This is comparable with what is seen in a strongly developing storm. The wavenumber of the fastest growing wave is $1.61f_0/ND = 1.61 \times 10^{-4}/(120) \text{m}^{-1} \approx 1.34 \times 10^{-6}\text{m}^{-1}$, giving a wavelength of $2\pi/k \approx 4700\text{ km}$. (At $45^0$, where a latitude circle measures $28000\text{ km}$, this corresponds to zonal wavenumber 6.)

The longitude-height structure of the most rapidly growing mode is shown in Fig. 4. Note:
Figure 4: [Holton Fig 8.10]

Properties of the most unstable Eady wave. (a) Contours of perturbation geopotential height; 
$H$ and $L$ designate ridge and trough axes, respectively. (b) Contours of vertical velocity; up 
and down arrows designate axes of maximum upward and downward motion, respectively. 
(c) Contours of perturbation temperature; $W$ and $C$ designate axes of warmest and coldest 
temperatures, respectively. In all panels 1 and 1/4 wavelengths are shown for clarity.
1. The geopotential perturbation maximizes at the upper and lower boundaries
2. The geopotential perturbation tilts westward with height
3. $w$ and $T$ are positively correlated
4. Poleward flow ($\partial \phi / \partial x > 0$) is positively correlated with $T$
Appendix: Solution of (7).

To clean up the eqs. a bit, temporarily use the shorthand $S = \sinh \left( \frac{1}{2} \kappa L \right)$, $C = \cosh \left( \frac{1}{2} \kappa L \right)$. Then rewrite (7) as

$$A \left[ \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) S - \Lambda C \right] + B \left[ \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) C - \Lambda S \right] = 0$$

\hspace{1cm} (10)

$$A \left[ \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) S - \Lambda C \right] + B \left[ -\frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) C + \Lambda S \right] = 0$$

Setting the determinant of coefficients to zero gives us

$$A \left[ \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) S - \Lambda C \right] \left[ -\frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) C + \Lambda S \right] - B \left[ \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) S - \Lambda C \right] \left[ \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) C + \Lambda S \right] = 0$$

Reorganizing,

$$-\kappa^2 \frac{L^2}{D^2} \left[ \left( \Lambda \frac{D}{2} \right)^2 - c^2 \right] SC + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) C^2 + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) S^2 - \Lambda^2 SC$$

$$-\kappa^2 \frac{L^2}{D^2} \left[ \left( \Lambda \frac{D}{2} \right)^2 - c^2 \right] SC + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) S^2 + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) C^2 - \Lambda^2 SC = 0 .$$

Then

$$2\kappa^2 \frac{L^2}{D^2} \left[ c^2 - \left( \frac{\Lambda}{2} \right)^2 \right] SC - 2\Lambda^2 SC + \Lambda^2 \kappa L \left( C^2 + S^2 \right) = 0 .$$

Now, using the identities

$$\frac{\cosh^2 x + \sinh^2 x}{\cosh x \sinh x} = 2 \coth 2x$$

and

$$\coth 2x = \frac{1}{2} (\tanh x + \coth x)$$
we arrive at
\[
c^2 = \left( \frac{\Lambda D}{2} \right)^2 - \frac{D^2 \Lambda^2}{L^2 \kappa^2} - \frac{\Lambda^2 D^2}{2\kappa L} \left[ \tanh \left( \frac{1}{2} \kappa L \right) + \coth \left( \frac{1}{2} \kappa L \right) \right]
\]
\[
= \left( \frac{\Lambda D}{\kappa L} \right)^2 \left[ \frac{\kappa^2 L^2}{4} - \frac{\kappa L}{2} \left( \tanh \left( \frac{1}{2} \kappa L \right) + \coth \left( \frac{1}{2} \kappa L \right) \right) - 1 \right]
\]
\[
= \left( \frac{\Lambda D}{\kappa L} \right)^2 \left[ \frac{\kappa L}{2} - \tanh \left( \frac{1}{2} \kappa L \right) \right] \left[ \frac{\kappa L}{2} - \coth \left( \frac{1}{2} \kappa L \right) \right] .
\]

(11)

This takes us directly to (8).