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# Turbulence Originating From Convectively Stable Internal Waves

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A theory has recently been developed for the generation of turbulence by unstable tides and gravity waves (Lindzen, 1981). Omitted from this theory was consideration of the possibility that turbulence could be generated by gravity waves which were not convectively unstable (i.e., breaking). The possibility of such generation was suggested by a result of McComas and Bretherton (1977) showing that internal gravity waves are indeed unstable to other gravity waves with shorter wavelengths. The present paper estimates the maximum turbulence which might be generated by such a process and shows that even this turbulence would not greatly alter earlier results.

## 1. INTRODUCTION

It is generally recognized that as internal gravity waves reach amplitudes for which they are unstable, they will break down, generating turbulence sufficient to inhibit further growth [*Hodges*, 1969; *Lindzen*, 1981]. The question arises as to whether smaller amplitude waves might generate turbulence as well. The analysis of *Weinstock* [1976] appears to suggest that this might be possible. (*Weinstock* [1976] has a criterion for saturation which does not appear to require breaking. We do not understand Weinstock's analysis sufficiently to comment further on it.) Apart from Weinstock's analysis, several studies suggest that the fields of gravity waves will cascade to higher wave numbers [*Dewan*, 1979; *McComas and Bretherton*, 1977]. In connection with this possibility, *McComas and Bretherton* [1977] showed that internal gravity waves were unstable to higher wave number gravity waves. (*McComas and Bretherton* [1977] study resonant triad interactions among gravity waves. However, one of their results concerning parametric subharmonic instability amounts to a proof that an internal gravity is unstable to a pair of internal gravity waves—each having $\frac{1}{2}$ the vertical wavelength of the original wave—one traveling upward, the other downward.) This strongly suggests that there will be some cascade to smaller scales. However, we have little further theory on this matter. Nevertheless, such cascade might lead to turbulence by generating waves with vertical wavelengths that are short enough to permit breaking. The resulting turbulence might prove important for both mixing and for attenuating the original sub-breaking gravity wave.

At present, we do not actually know whether convectively stable gravity waves cascade so as to form waves at scales where breaking can occur. Despite this ignorance, we will attempt to show, in the present brief paper, that such cascades, if they do exist, will not significantly modify earlier results that omitted such cascades [*Lindzen*, 1981]. The additional eddy diffusion attributable to such cascade will be shown to be small, compatible with current observations. Our approach is to assume the existence of a cascade and to specifically consider a cascade wherein smaller scales have the same amplitudes as larger scales (i.e., cascade without loss of amplitude). Such a cascade is intuitively more effective than any real cascade is likely to be. The fact that the eddy diffusion resulting from even this cascade model is still of very limited importance allows us to reach the above conclusions despite our ignorance.



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## 2. CASCADE MODEL

Dimensionally, we expect vertical eddy diffusion coefficients to be of the form

$$D \sim W' \cdot h \tag{1}$$

where $W'$ is a characteristic eddy vertical velocity and $h$ is a characteristic vertical scale. Of course, internal gravity waves have vertical velocity amplitudes which are simply related, at least approximately, to more readily measured temperature amplitudes through the relation

$$W' \approx \frac{\omega \Upsilon'}{\Gamma} \tag{2}$$

where $\omega$ = frequency (Doppler shifted when mean flow is present), $\Upsilon'$ temperature amplitude, and $\Gamma = d\bar{\Upsilon}/dz + q/c_p$ where $\bar{\Upsilon}$ = mean temperature [see *Lindzen*, 1981]. In addition, in the absence of shear in the mean flow, gravity waves are associated with two vertical scales: $\lambda^{-1}$ (where $\lambda$ = vertical wave number) and $2H$ (where $H$, here, refers to the pressure scale height), associated with growth of amplitude with height. For simplicity we will, in this note, ignore effects due to shear in the basic flow. Now, one might conceivably use the above magnitudes in (1) to obtain

$$D \sim f(\lambda H) W' \lambda^{-1} \tag{3}$$

but, in general, we do not expect a gravity wave (or even the superposition of many gravity waves) to stir the environment [*Andrews and McIntyre*, 1976]. An exception to this is a breaking gravity wave where

$$\lambda \Upsilon' \sim \Gamma \tag{4}$$

(Equation (4) assumes $\lambda \gg 1/2H$. If $\lambda < 1/2H$, then $\lambda$ should be replaced by $1/2H$.) In this case, *Hodges* [1969] and more recently *Lindzen* [1981] have shown that

$$D \approx \frac{1}{2H\lambda} \frac{\omega T'^2}{\Gamma^2} \tag{5}$$

(In the limit of no mean flow, *Lindzen* [1981] actually obtains

$$D = \frac{1}{2H\lambda} \frac{\omega}{\lambda^2}$$

at breaking. Substitution of (4) leads to (5).) In the absence of breaking, one might expect $D = 0$. However, if due to nonlinear cascade vertical scales smaller than $\lambda^{-1}$ are generated, then it is conceivable that despite smaller amplitudes, $\lambda$ might be sufficiently large to cause breaking at the smaller scale and, therefore, diffusion.

Unfortunately, at present, there is limited direct observational or theoretical evidence for such a cascade [*van Zandt*, 1982], though some theoretical arguments have been put forth [*Dewan*, 1979]. However, some interesting insights can be obtained by hypothesizing a cascade and investigating the consequences.

Assuming our basic wave is sub-breaking with amplitudes $\Upsilon$, $W = \omega\Upsilon/\Gamma$, and vertical wavelength $\lambda$. We will assume that at some larger $\lambda'$,

$$T_{\lambda'} \sim \Upsilon\left(\frac{\lambda}{\lambda'}\right)^{\alpha} \tag{6}$$

and

$$W_{\lambda'} \sim \frac{\omega\Upsilon}{\Gamma}\left(\frac{\lambda}{\lambda'}\right)^{\beta} \tag{7}$$

where $\alpha$ and $\beta$ are for the moment assumed cascade laws. If $\alpha < 1$, then there will exist a convectively unstable (breaking) $\lambda'$ where

$$\lambda'\Upsilon_{\lambda'} \sim \Upsilon\lambda^{\alpha}\lambda'^{1-\alpha} \sim \Gamma$$

i.e.,

$$\lambda' \sim \left(\frac{\Gamma}{T_{\lambda}\alpha}\right)^{1/1-\alpha} \tag{8}$$

For the breaking scale we might expect

$$D \approx f(\lambda H) W_{\lambda'} \frac{1}{\lambda'}$$

$$\approx f(\lambda H) \frac{\omega\Upsilon^2}{\Gamma^2}\left(\frac{\lambda\Upsilon}{\Gamma}\right) \tag{9}$$

If we further require that when $\lambda' = \lambda$, (9) reduces to (5), then we take

$$f(\lambda H) = \frac{1}{2H\lambda} \tag{10}$$

(A priori, it might seem more reasonable to use $1/2H\lambda'$. However, this form assumes that $D$ is exactly sufficient to prevent growth with height, the cause of breaking for the primary wave. For $\lambda' > \lambda$, the direct cause of breaking is cascade instead. Moreover, at small scales we expect dependence on $H\lambda'$ to become asymptotically unimportant. Thus, the choice in (10) seems more appropriate.) and

$$D \approx \frac{1}{2H\lambda} \frac{\omega T^2}{\Gamma^2}\left(\frac{\lambda T}{\Gamma}\right)^{\alpha+\beta/1-\alpha} \tag{11}$$

The maximum value of $D$ does correspond to $\lambda\Upsilon = \Gamma$, in which case

$$D = D_{\max} = \frac{1}{2H\lambda} \frac{\omega}{\lambda^2} \tag{12}$$

(11) can be rewritten

$$D \approx D_{\max}\left(\frac{\lambda T}{\Gamma}\right)^{n} \tag{13}$$

where

$$n \equiv \frac{2 - \alpha + \beta}{1 - \alpha} \tag{14}$$

The choice of $\alpha$ and $\beta$ is of course our major problem. Indeed, as was already mentioned, the existence of $\alpha$ and $\beta$ is even in question. However, it seems reasonable to assume that even if there is cascade, $\alpha$ and $\beta$ are unlikely to be less than zero. Otherwise, cascade products would dominate the original waves, and flow fields would always be dominated by breaking waves. This is certainly contrary to observations. Thus results for $\alpha = \beta = 0$ should provide an upper bound on the turbulence resulting from the cascade of sub-breaking waves. For $\alpha = \beta = 0$, $n = 2$. Thus, even in this case $D < D_{\max}$ for $|\lambda T| < \Gamma$. This is because the 'mixing length' associated with breaking is markedly reduced.

It is interesting to note that a $\lambda^{-5/3}$ power spectrum would lead to $\alpha = \beta = 1/3$ and $n = 3$. As we shall see, the consequences of such a choice do not differ very much from the results for $n = 2$.

## 3. Some Estimates

Before proceeding to our main question (namely, whether turbulence generated via cascade could significantly attenuate the original wave, perhaps even preventing its breaking), it may prove interesting to use our formulae in order to check some orders of magnitude.

### 3.1. *Turbulent Diffusion in the Tropical Stratosphere*

Short period wave motions in the tropical lower stratosphere appear to be dominated by mixed gravity Rossby waves and by Kelvin waves [*Lindzen and Tsay*, 1975]. Typical values associated with these waves and with the lower stratosphere are

$$\Upsilon \sim 3°\text{C}$$

$$\Gamma \sim 11°/\text{km}$$

$$\omega \sim 2\pi/10 \text{ days}$$

$$2\pi/\lambda \sim 10 \text{ km}$$

$$H \sim 7 \text{ km}$$

Substituting these values in (13) and taking $n = 2$ we get

$$D_{\max} \approx \frac{1}{2H\lambda} \frac{\omega}{\lambda^2} \approx 2.1 \frac{m^2}{s}$$

and

$$D \approx D_{\max}\left(\frac{\lambda T'}{\Gamma}\right)^2 \approx 6.15 \times 10^2 \text{ cm}^2/\text{s} \tag{15}$$

Equation (15), given our cascade assumption ($\alpha = 0 = \beta$), is almost certainly an overestimate. It is interesting, therefore, to note that the value given in (15) is quite modest. Current estimates are that $D \ll 1 - 5 \times 10^2$ cm²/s, not far from (15) [*Lilly et al.*, 1974]. The above estimate for $D$ is much smaller than those derived from one-dimensional models for chemical composition [*Wofsy and McElroy*, 1973]. However, the latter estimates included transport due to planetary waves and mean circulations, not just turbulence. Note that

$$\left|\frac{\lambda T}{\Gamma}\right| \approx 0.17$$

so that the observations are compatible with a choice of $n$ between 2 and 3.

It has been argued that sources of turbulence other than wave cascade exist in the lower stratosphere. A frequently cited example is wave breaking due to wave amplification in the neighborhood of critical layers [*Geller et al.*, 1975]. Such a mechanism is likely to occur in only limited regions and is likely to generate only weak diffusion [*Lindzen*, 1981]. It is, therefore, far from obvious that cascade is not a competitive possibility.

### 3.2. *Relative Importance of Diurnal and Semidiurnal Tides in the Generation of Turbulence*

A major part of the ambient wave field in the tropical mesosphere is due to tides: diurnal ($\omega = 2\pi$/day) and semidiurnal ($\omega = \pi$/day) [viz *Chapman and Lindzen*, 1970; *Lindzen*, 1979]. From (12) we see that $D_{max}$ is proportional to $\omega$ and inversely proportional to $\lambda^3$, and thus strongly favors the semidiurnal mode. ($\lambda \approx 1/3.5$ km for the main propagating diurnal tidal mode; for the main semidiurnal mode, $\lambda$ is so small that it is replaced by $1/2H$. As an aside, (12) implies that the maximum $D$ producible by any internal wave will be limited to $D_{supermax} \approx (1/2H)(\omega_B/1/(2H)^3)$, where $\omega_B$ = Brunt-Vaisala frequency. $D_{supermax} \approx 4.7 \times 10^6$ m²/s. These results are, of course, independent of assumed cascades.)

However, near 80 km the diurnal mode is believed to be unstable, while the semidiurnal is not. From (4) we see that this is associated with

$$T'_{diur} \sim 30°\text{K}$$

and

$$D \approx D_{max} \approx \frac{1}{2H(1/3.5 \text{ km})^3} \frac{2\pi}{86{,}400 \text{ s}} \approx 2 \times 10^2 \frac{\text{m}^2}{\text{s}}$$

consistent with the results of *Lindzen* [1981].

The semidiurnal tide at this height is generally believed to be associated with $T'_{semidiur} \sim 10°$K and from (13) we get (taking $n = 2$)

$$D \approx 1.2 \times 10^2 \text{ m}^2/\text{s}$$

Despite our assumption of unrealistically efficient cascade, this is still less than diffusion from the breaking diurnal tide. However, above 95 km, it is possible that the semidiurnal mode might be the major source of diffusion and might, in fact, significantly attenuate the propagating diurnal mode.

## 4. SELF-CONSISTENT WAVE BREAKING CALCULATIONS

Once one considers the possibility that stable, nonbreaking internal waves can generate turbulence, the question arises as to whether this turbulence might actually prevent the breaking of the primary wave. We have chosen to study this question in the context of the main propagating diurnal mode. Using the EGM (equivalent gravity mode) formalism of *Lindzen* [1970], one may calculate the vertical structure of the main propagating mode forced by daily variations in insolation absorption by $O_3$ and $H_2O$ in the presence molecular viscosity (indicated by the dashed-dotted curve in Figure 2). The resulting vertical structure is shown by the solid curve in Figure 1. The solid curve indicates breaking above about 82 km. For the following calculations we have taken $n = 3$. Equation (13) gives an eddy diffusion coefficient associated with the solid curve as shown by the solid curve in Figure 2. One may next recompute the vertical structure by using the sum of this eddy diffusion and molecular diffusion, recompute the eddy diffusion, and keep on repeating this whole procedure until convergence is obtained. The result is shown by the dashed curve in Figure 1. Below 80 km there is little difference between the initial and the converged solutions. Above 80 km the converged solution is, of course, smaller, but breaking still occurs, albeit at a somewhat greater altitude. Consistent with the above is the fact that converged eddy diffusions are about the same as the initial determination below 80 km, but somewhat less between 80 km and the final breaking level (viz Figure 2). Interestingly, the converged eddy diffusion above 80 km is very insensitive to the choice of cascade law (viz Figure 3). Thus, the particular

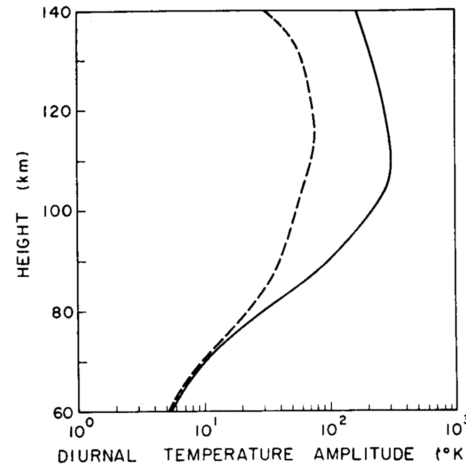


Fig. 1. Equatorial diurnal temperature amplitudes for zeroth iteration (solid curve) and converged solution (dotted curve) corresponding to $n = 3$ power law.

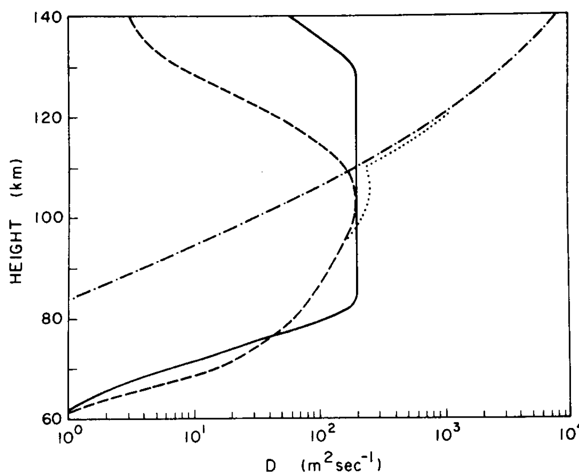


Fig. 2. Profiles of eddy diffusion coefficient for zeroth iteration (solid curve) and converged solution (dashed curve) corresponding to $n = 3$ power law, profile of molecular diffusion coefficient (dashed-dotted curve), and sum of molecular and converged eddy diffusion coefficient profiles (dotted curve).

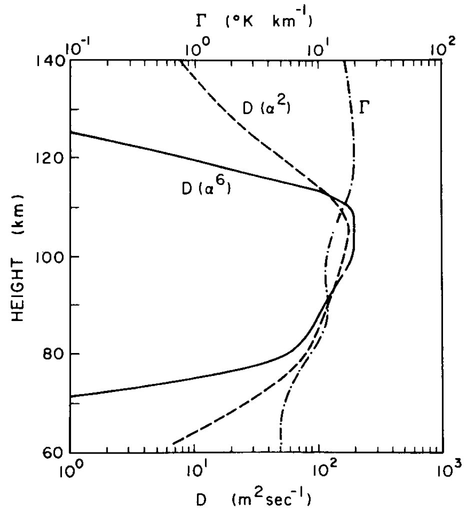


Fig. 3. Stability function (Γ) and converged eddy diffusion coefficient profiles corresponding to $n = 2$ (dashed curve) and $n = 6$ (solid curve) power laws.

choice of $n$ in Figures 1 and 2 is not important. Below 80 km, eddy diffusion depends strongly on the assumed cascade law; however, at these levels the eddy diffusion has little effect on the tide itself. Note that eddy diffusion above 110 km is irrelevant insofar as molecular diffusion is much larger.

## 5. Observational Possibilities

The above calculations are able to bound the role of cascade to shorter vertical wavelengths without in any way demonstrating that such a process actually exists. However, as observational methods with high vertical resolution becomes available, the possibility of observational confirmation or rejection becomes possible. Assuming $\alpha > 0$ and $\beta > 0$, (4) tells us

$$\lambda' \gtrsim \frac{\Gamma}{T} \tag{16}$$

Evaluation of (16) allows some estimate of the amount of resolution which would be necessary.

For the parameters in section 3.1, we get

$$\lambda' \gtrsim \frac{11°/\text{km}}{3°} \cong 3.7 \text{ km}^{-1}$$

or vertical wavelength $\lesssim (2\pi/3.7 \text{ km}) - 1 \cong 1.7$ km. Thus, in the tropical stratosphere, resolution better than 1.7 km/$2\pi \approx 270$ m would be needed.

Table 1 gives similar estimates for the main propagating diurnal mode. It is evident that breaking due to cascade of this mode will be very difficult to observe below 50 km.

TABLE 1. Maximum Wavelengths Associated With Characteristic Amplitudes of the Main Propagating Diurnal Tide at Different Altitudes, $Z$ (Assuming $n = 2$)

| $Z$, km | $\delta T$, °C | Lower Bound for $\lambda'$, km$^{-1}$ | Upper Bound for $2\pi/\lambda'$, km |
|---|---|---|---|
| 30 | 0.5 | 22 | 0.28 |
| 40 | 1 | 11 | 0.57 |
| 50 | 2 | 5.5 | 1.1 |
| 65 | 5 | 2.2 | 2.9 |
| 75 | 10 | 1.1 | 5.7 |

The remaining question is how one distinguishes turbulence from waves. Two characteristics may be amenable to observation:

1. Internal waves are characterized by horizontal scales which are much larger than vertical scales, whereas, for convective instabilities, horizontal and vertical scales are comparable. Thus, as one observes smaller and smaller vertical scales, one should begin to see a breakdown in horizontal coherence as $\lambda$ exceeds $\lambda'$.
2. Similarly, convective elements are generally associated with shorter time scales than are internal waves. Presumably, a time-vertical space spectral analysis could reveal such a transition.

Finally, if independent estimates of $D$ are available, measurements of temperature variance, characteristic frequency, and vertical scales should permit the determination of whether a cascade parameter exists and if so what its value is.

## 7. Concluding Remarks

Underlying the present conjectural work is the notion introduced in *Lindzen et al.* [1980] that turbulent mixing can only arise from mass redistributing instabilities such as convective and/or inertial instability. Thus, for stable internal waves to generate turbulence there must first be a transfer of energy into convectively unstable scales. In this paper we have put forth a simple parameterization for such a cascade. Despite the ad hoc nature of the parameterization, it permits one particular concrete result to be developed: namely, that turbulence due to stable waves does not diminish the amplitude of waves below the level at which they would have broken in the absence of turbulence. It should be noted that our 'worst case' cascade ($\alpha = \beta = 0$) would itself strongly modify the basic wave regardless of ultimate breaking. Observations of most internal gravity waves do not have sufficient time and height resolution to discount this possibility unambiguously. However, observations of tides (summarized in *Chapman and Lindzen* [1970] and *Lindzen* [1979]) do display the characteristics of the main modes including the anticipated growth with height, at least below 80 km. Thus, cascade for these modes is almost certain to be less effective than our 'worst case.'

In addition to a variety of speculative estimates, we suggest some ways by which observations can help make our ideas of how internal waves generate turbulence more concrete. Finally, it should be noted that the sudden onset of diffusion at the breaking level as proposed in *Lindzen* [1981] can cause difficulties when incorporated into numerical models of the large scale mesospheric circulation. Equation (13) provides a plausible expression for smoothing the onset, and section 4 suggests that this smoothing will not greatly affect the primary wave. The deposition of momentum will, however, be spread somewhat.

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