

## Instabilities in a Stratified Fluid Having One Critical Level. Part II: Explanation of Gravity Wave Instabilities Using the Concept of Overreflection

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### ABSTRACT

We show that there is a one-to-one correspondence between the values of the parameters  $Ri$  (Richardson number at the critical level, where mean flow equals phase speed),  $k$  (horizontal wavenumber) and  $c_r$  (phase speed) for which gravity wave instabilities in the presence of a rigid lower boundary were found in Part I and the parameters  $Ri$ ,  $k$  and  $c_r$  for which overreflection of a neutral wave incident on the shear zone from below would occur. A simple formula involving the reflection coefficient, vertical group velocity of the mode, and the ground to shear layer distance can give quantitative estimates of the growth rate that are especially accurate for small growth rates. The maximum growth rates and corresponding reflection coefficients can be parameterized in a more physically meaningful manner in terms of the Richardson number at the critical level than in terms of the minimum Richardson number in the background wind profile.

### 1. Introduction

In this part we show that there is a one-to-one correspondence between the values of the parameters  $Ri$  (Richardson number at the critical level, where mean flow equals phase speed),  $k$  (horizontal wavenumber) and  $c_r$  (phase speed) for which gravity wave instabilities in the presence of a rigid lower boundary were found previously in Part I and the values of the parameters  $Ri$ ,  $k$  and  $c_r$  for which overreflection of a neutral wave incident on the shear zone from below would occur. As explained by Lindzen and Rosenthal (1976), this is because a wave packet propagating between a ground and shear layer, which is totally reflected each time it hits the lower boundary and overreflected at each interaction with the shear layer, will show an apparent increase in amplitude with time, or instability—provided that incident and reflected waves are in phase. In particular, we now find that overreflection of waves propagating above the shear layer can occur only for Richardson numbers less than about 0.116, a conclusion similar to that reached by Jones (1968), Eltayeb and McKenzie (1975) and Lindzen and Tung (1976). Also, overreflection for waves evanescent above the shear zone can occur for all Richardson numbers less than 0.2499. We show that a simple formula involving the reflection coefficient, vertical group velocity of the mode and ground to shear layer distance can give quantitative estimates of the growth rate that are especially accurate for small growth rates.

### 2. Model and equations

We now derive the expression for the reflection coefficient,  $R_w$ , which will enable us in Section 3 to show that gravity wave instabilities occur only when the neutral wave having the same phase speed  $c_r$  as the instability is overreflected by the critical level and returned to it by a lower turning point or the ground. With the same notation as used in Part I on instabilities, we consider a neutral upward propagating wave of the form

$$w_i = A_i \exp[-n_B(z + \Delta H) + k(x - ct)], \quad (1)$$

incident on the shear layer, where we assume for now that  $k$  and  $c$  are such that

$$n_B = \{k^2 - [N/(U + c)]^2\}^{1/2}$$

is purely imaginary, with the choice  $\text{Im}n_B \geq 0$  ensuring that  $w_i$  given by (1) represents an incident wave carrying energy toward the shear layer. Since  $\text{Re}n_B = 0$ , we are restricted to regions A or C of  $(k, k\hat{c}_r)$  space as shown in Fig. 2 of Part I. Now, the reflected wave below the shear zone may be written as

$$w_r = A_r \exp[n_B(z + \Delta H) + k(x - ct)],$$

and the transmitted disturbance above the shear zone is

$$w_t = A_t \exp[-n_H(z - \Delta H) + k(x - ct)],$$

where  $n_H$  is the square root of  $\{k^2 - [N/(U - c)]^2\}^{1/2}$

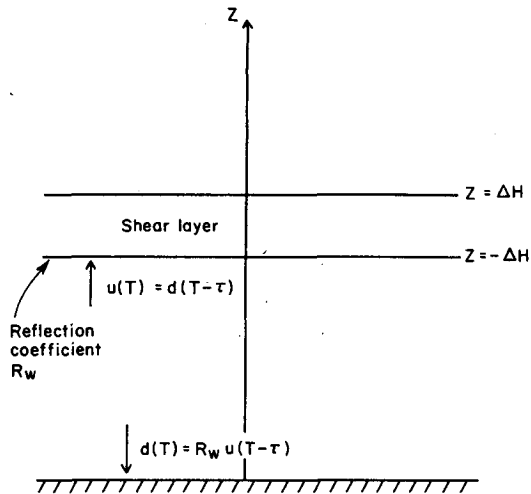


FIG. 1. Mechanism by which overreflection of wave packets leads to instability in models having one critical level.  $u(T)$  is the amplitude of an upward propagating packet below the shear layer at time  $T$ , just before it is incident on the shear layer and  $d(T)$  is the amplitude of a downward propagating packet at time  $T$ , just before it hits the rigid lower boundary.

with  $\text{Re}n_H \geq 0$  and  $(\text{Im}n_H)(c_r - U) \geq 0$  (by the radiation condition).

The use of Eqs. (16) and (17) from Part I shows that the reflection coefficient at  $z = -\Delta H$  is given by

$$R_w = \left| \frac{A_r}{A_i} \right| = \left| \frac{h_2}{h_1} \right|, \quad (2)$$

where  $h_1$  and  $h_2$  are defined in Eqs. (22) and (28) in Part I. We note that  $R_w$  goes to infinity if and only if  $h_1 = 0$ , which is precisely the dispersion relation [Eq. (21) in Part I] for an unbounded fluid with one critical level. Thus, the existence of internal gravity wave solutions propagating energy away from one side of a shear zone in an unbounded fluid (as was found in Part I) is equivalent to the existence of infinite overreflection (i.e., outgoing energy with no incoming energy) by a critical level with nonzero Richardson number. This extends the concept of equivalence of infinite overreflection and neutral solutions in an unbounded fluid noted by Lindzen (1974) and Lindzen and Rosenthal (1976) for zero Richardson number critical layers to critical layers with nonzero Richardson number.

### 3. Instability and overreflection

In the same manner as for shear zones with zero Richardson numbers (Lindzen and Rosenthal, 1976), we now show that overreflection leads to instability in the presence of a rigid lower boundary for shear zones with nonzero Richardson numbers. The mechanism involved is that a wave overreflected by a factor  $R_w$  at the shear layer is totally reflected at the lower boundary and returned to the shear layer to be further

overreflected, resulting in an apparent growth rate  $kc_i$ . As noted by Lindzen and Rosenthal (1976), if the wave takes a time  $\tau$  to travel from the bottom of the shear layer at  $z = -\Delta H$  to the ground ( $z = -H$ ) and instability is due to overreflection, we expect

$$R_w \approx \exp(2kc_i\tau). \quad (3)$$

Although it is not necessary for this section, we now present a detailed derivation of (3) using a formalism that is useful because it can be extended readily to models with several critical levels and a rigid boundary present. (Another overreflecting critical layer below a first shear layer would be expected to give gravity waves higher growth rates than just a rigid boundary below a shear layer.) Let  $u(T)$  represent the amplitude of an upward propagating wave train below the shear layer at time  $T$ , just before it is incident on the shear layer, and let  $d(T)$  be the amplitude of a downward propagating wave train at time  $T$ , just before it hits the rigid lower boundary, as shown in Fig. 1. Then, if the reflection coefficient at the bottom of the shear layer is  $R_w$  and the propagation time for waves travelling from the shear layer's bottom to the ground is  $\tau$ , we have

$$\left. \begin{aligned} u(T) &= d(T - \tau) \\ d(T) &= R_w u(T - \tau) \end{aligned} \right\}, \quad (4)$$

which imply

$$\left. \begin{aligned} u(T) &= R_w u(T - 2\tau), \\ d(T) &= R_w d(T - 2\tau). \end{aligned} \right\}$$

These equations may be solved by inspection, or by using standard methods for solving simple difference equations (e.g., p. 323 of Conte and de Boor, 1972) to give

$$\left. \begin{aligned} u(T) &= [R_w^{T/(2\tau)}]u(0) = u(0) \exp[T(\ln R_w)/(2\tau)] \\ d(T) &= [R_w^{T/(2\tau)}]d(0) = d(0) \exp[T(\ln R_w)/(2\tau)] \end{aligned} \right\}, \quad (5)$$

which results in an apparent growth rate  $kc_i$  given by (3).

Finally, in order to estimate growth rates using reflection coefficients, we need to calculate the propagation time  $\tau$  for waves to travel from the bottom of the shear layer to the lower boundary. As a first approximation, we assume that  $c_i/U$  is sufficiently small that the main properties of the unstable mode, such as its vertical group velocity  $v_g$  and vertical wavenumber  $m_B$  are the same as for the neutral mode with the same  $k$  and  $c_r$ . Then, the vertical group velocity  $v_g$  below the shear layer is given by

$$v_g = \frac{\partial(kc)_r}{\partial m_B} = -\frac{km_B N}{(m_B^2 + k^2)^{3/2}} = -\frac{km_B(U + c_r)^3}{N^2}, \quad (6)$$

where  $m_B = \text{Im}n_B > 0$ , and the propagation time  $\tau$  is

$$\tau = \frac{H - \Delta H}{|v_g|} = \frac{2HN^2}{km_B(U + c_r)^3} \quad (7)$$

Eqs. (2) and (3) can be used to predict growth rates from overreflection coefficients, as done by Lindzen and Rosenthal (1976) for a Helmholtz velocity profile.

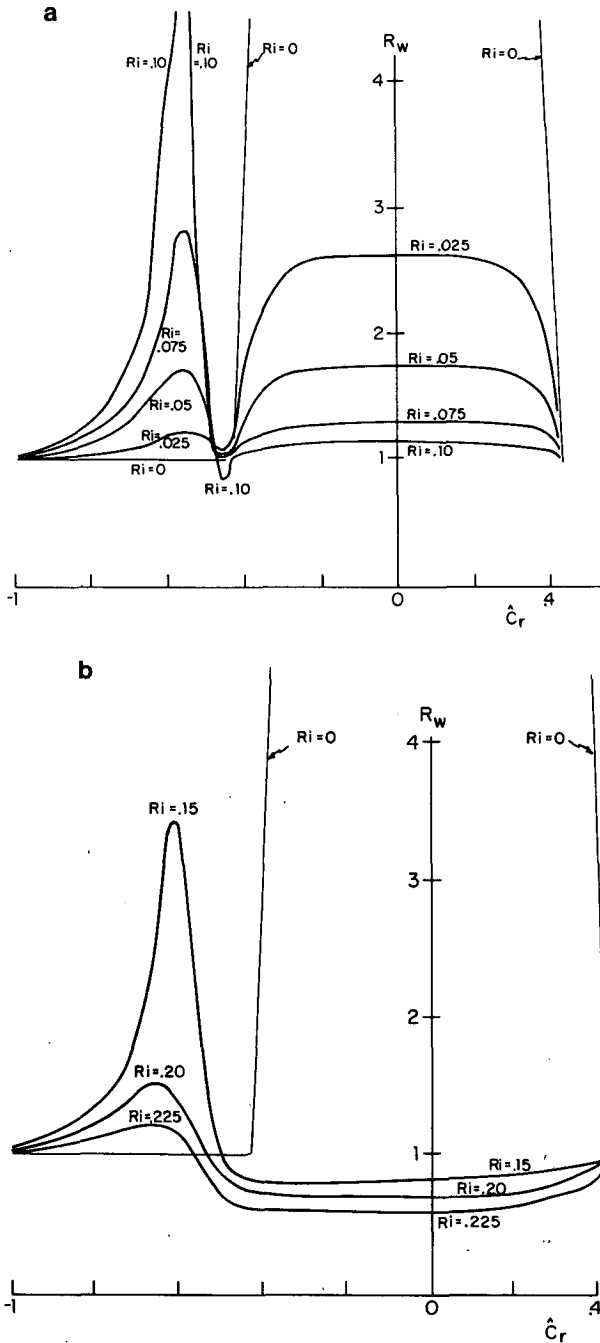


FIG. 2. Reflection coefficient  $R_w$  versus phase speed  $\hat{c}_r$  for a horizontal wavenumber  $\hat{k} = 0.7$ , with  $Ri$  as a parameter: (a)  $Ri = 0-0.1$ , (b)  $Ri = 0.15-0.225$ . Also shown is the curve for  $Ri = 0$ .

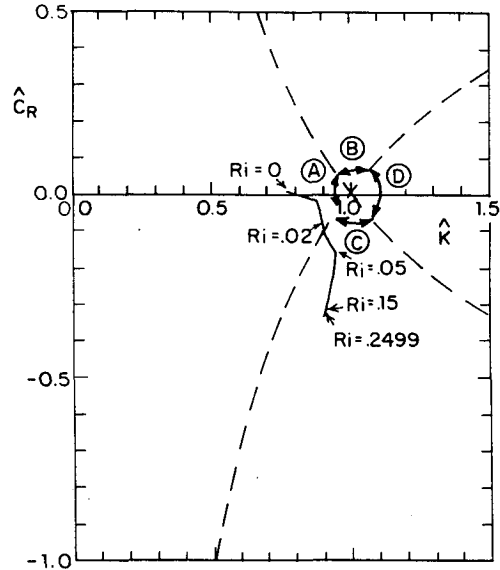


FIG. 3. Locus of the values of  $(\hat{c}_r)_{disp}$ , corresponding to the gravity waves with the largest dispersion growth rate for all  $\hat{H}$  and  $\hat{k}$ , in  $(\hat{k}, \hat{c}_r)$  space as the parameter  $Ri$  is varied. Also shown are the regions of  $(\hat{k}, \hat{c}_r)$  space discussed in the text.

As a qualitative check on the correlation between instability and overreflection for a velocity profile with nonzero Richardson number ( $Ri$ ) in the shear layer, we plot the reflection coefficients  $R_w$  versus phase speeds  $\hat{c}_r$  in Figs. 2a and 2b for a horizontal wavenumber  $\hat{k} = 0.7$ , a value nearly equal to that found by Lindzen (1974) for the "most efficient" neutral wave, and a value fairly close to those typical of unstable gravity waves when a lower boundary is present. Fig. 2a and similar figures not shown here of  $R_w$  versus  $\hat{c}_r$  with  $Ri$  as a parameter show that for  $0 \leq Ri \leq 0.03$ , the largest reflection coefficients occur in region A of  $(\hat{k}, \hat{c}_r)$  space, where waves propagate vertically both above and below the shear layer. This is consistent with Fig. 3 (obtained using results from Part I), which shows that for small  $Ri$ , the most unstable gravity waves are located in region A of  $(\hat{k}, \hat{c}_r)$  space. Fig. 2a shows that for  $0.03 \leq Ri \leq 0.116$ , there is overreflection both in region A and in region C of  $(\hat{k}, \hat{c}_r)$  space, with the maximal reflection coefficients now occurring in region C, where solutions propagate vertically below the shear layer but decay above it. This agrees with Fig. 3 and graphs such as Fig. 5 of Part I, which show that for this intermediate range of  $Ri$ , the most unstable gravity waves are found in region C of  $(\hat{k}, \hat{c}_r)$  space, although instabilities still are found in region A of  $(\hat{k}, \hat{c}_r)$  space.

As shown in Fig. 2b and graphs such as Fig. 7 of Part I, when the Richardson number in the shear layer exceeds 0.116, there is no overreflection of gravity waves that would propagate vertically above the shear layer, and no instabilities of such waves in region A of  $(\hat{k}, \hat{c}_r)$  space can be found. Nevertheless, we

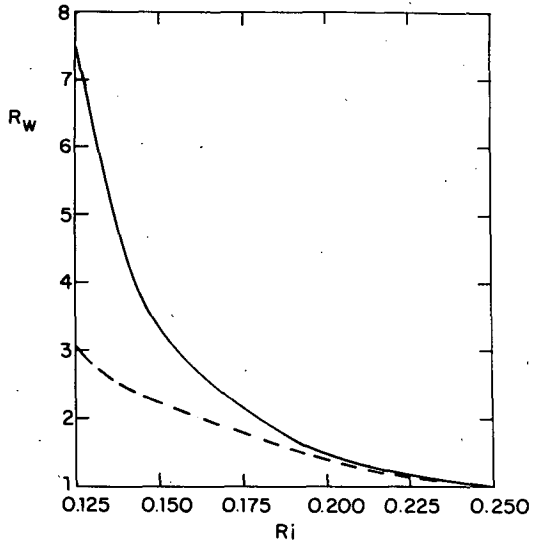


FIG. 4. Values of  $R_{\text{eff}}$  (dashed line) occurring at the values of  $(\hat{c}_r)_{\text{disp}}$ ,  $\hat{H}_g$  and  $\hat{k}_g$  at which the maximal dispersion growth rates for gravity waves occur, as a function of the Richardson number at the critical level. Shown with a solid line are the values of  $R_w$  that occur at the same  $(\hat{c}_r)_{\text{disp}}$ ,  $\hat{H}_g$  and  $\hat{k}_g$ .

find overreflection of gravity waves that are evanescent above the shear layer for all Richardson numbers less than 0.2499. This is consistent with the presence of gravity wave instabilities in region C of  $(\hat{k}, \hat{c}_r)$  space for all Ri up to 0.2499 (which is very close to the theoretical upper bound of  $1/4$  established by Miles, 1961), as shown in Fig. 3. Finally, we note that Figs. 2a and 2b show overreflection occurs for gravity waves in region C of  $(\hat{k}, \hat{c}_r)$  space for all nonzero Richardson numbers less than  $1/4$ , whereas Lindzen and Rosenthal (1976) found that gravity waves would only be totally reflected in that region for a Helmholtz velocity profile with zero Richardson number at the shear level.

In order for reflection coefficients to predict growth rates accurately, the effective reflection coefficient,  $R_{\text{eff}}$  (inferred from the dispersion growth rate,  $k(c_i)_{\text{disp}}$ , found using (27) in Part I) given by

$$R_{\text{eff}} = \exp[2k(c_i)_{\text{disp}}\tau] \quad (8)$$

should be consistent with the reflection coefficient  $R_w$  calculated with (2). To check whether this is the case, we plot in Fig. 4 the values of  $R_{\text{eff}}$  corresponding to the values of  $(\hat{c}_r)_{\text{disp}}$ ,  $\hat{H}_g$  and  $\hat{k}_g$  (shown in Figs. 14 and 15 in Part I) at which the maximal values of  $\hat{k}(\hat{c}_i)_{\text{disp}}$  (shown in Fig. 16 in Part I) occur and compare these values of  $R_{\text{eff}}$  to the values of  $R_w$  found using (2) at  $(\hat{c}_r)_{\text{disp}}$ ,  $\hat{H}_g$  and  $\hat{k}_g$ . We find that  $R_{\text{eff}}$  and  $R_w$  obtained in this manner agree well when  $R_w < 2$ , which is the case when  $0.18 < \text{Ri} < 0.25$ . Thus, at the larger  $R_w$  possible when  $\text{Ri} < 0.18$ , either a more refined method of finding the reflection coefficient  $R_w$

and travel time  $\tau$  than that given by (2) and (7), respectively (which were derived for neutral waves), or else a method of explaining why large values of  $R_w$  cannot be attained in the limited time available for their development as waves travel from the shear zone to the ground and back is necessary. [Both methods of explaining the discrepancy in  $R_{\text{eff}}$  and  $R_w$  apparent in Fig. 4 as Ri decreases towards 0.125 were considered by Rosenthal (1981). A paper is in preparation in which the effect of the finite rate at which overreflection develops is computed using results of McIntyre and Weissman (1978).]

Before demonstrating the predictive power of reflection coefficients in estimating growth rates, we note that the consistency of the parameters giving maximal reflection coefficients with those giving maximal dispersion growth rates was also illustrated by Davis and Peltier (1977) for a tanh-background wind profile. Consideration of Fig. 7 used by Davis and Peltier (1977) is useful in answering the question as to whether reflection coefficients are characteristic of the minimum Richardson number in the background wind profile or of the Richardson number at the level where the waves have their critical level. For the tanh-velocity profile, with a minimum Richardson number  $J = 0.125$  occurring at the profile's inflection point, considered by Davis and Peltier (1977), waves which propagate above the shear zone are not overreflected, as shown by our Fig. 2. This applies, in particular, to waves with critical levels at the inflection point, where the local Richardson number is 0.125. The only waves that are overreflected at this value of  $J$  exceeding 0.116 decay above the shear zone (as mentioned earlier), which results in their critical levels being far enough below the inflection point that the Richardson number appropriate for their critical level is significantly larger than 0.125. Indeed, for the values of  $k$  and  $c_r$  at which the maximal  $R_w$  (apparently between 3 and 4) occurs in Fig. 7 of the work by Davis and Peltier (1977), a simple calculation shows that the Richardson number at the critical level is 0.1528. Our Fig. 4 shows that  $\text{Ri} = 0.1528$  is characterized by  $R_w = 3.2$  whereas  $\text{Ri} = 0.125$  is characterized by  $R_w = 7.5$ . Thus, the Richardson number at the critical level is a more relevant parameter in determining reflection coefficients than the minimum Richardson number in the background wind profile, and our method of presenting results as a function of the Richardson number at the critical level is a physically meaningful improvement to the presentation in terms of the Richardson number at the inflection point height, as given by Davis and Peltier (1977). (The relevance of the Richardson number at the critical level is also shown implicitly by its appearance in the theoretical work of Booker and Bretherton, 1967; Miles, 1963.)

To demonstrate how well reflection coefficients can quantitatively predict growth rates, we should, for

specified  $k$  and  $c_r$ , first determine an appropriate ground to shear layer spacing  $H$  from the quantization condition that the phase of  $(-A_r/A_i)$  given by  $\exp[-2\hat{n}_B(\text{Ri}^{1/2} - \hat{H})]$  [as can be found from the condition that  $w = 0$  at the ground, as stated in Eq. (18) in Part I] should equal the phase of  $(-A_r/A_i)$  given by  $(-h_2/h_1)$  [as can be derived from Eqs. (16) and (17) in Part I]. Henceforth, we denote the phase of  $(-h_2/h_1)$  by  $P$ , with  $-\pi < P \leq \pi$ . Then, our quantization condition can be written as

$$-2(\text{Im}\hat{n}_B)(\text{Ri}^{1/2} - \hat{H}) = P + 2M\pi,$$

(where  $M$  is an integer), which in turn implies that

$$(\text{Im}\hat{n}_B)(\hat{H} - \text{Ri}^{1/2})\pi^{-1} = M + \frac{1}{2\pi} P, \quad (9)$$

$$\hat{H} = \text{Ri}^{1/2} + \frac{\pi}{\text{Im}\hat{n}_B} \left( M + \frac{1}{2\pi} P \right). \quad (10)$$

Physically, the quantization condition given by (9) implies that the number of vertical wavelengths in the distance travelled between successive encounters with the overreflecting shear layer (viz.  $2(\hat{H} - \text{Ri}^{1/2})$ ) can have only discrete values separated by integers.

Now, given any  $\hat{k}$  and  $\hat{c}_r$ , we can determine a value of  $\hat{H}$  for which quantization will be satisfied using (10) and proceed to predict the growth rate  $k(c_i)_{\text{pred}}$  using

$$k(c_i)_{\text{pred}} = [\ln(R_w)]/(2\tau) \quad (11)$$

with (2) and (7). We can then compare this value with the value  $k(c_i)_{\text{disp}}$  found with the dispersion relation (Eq. (27) in Part I). To predict the growth rates of the most unstable gravity waves, we evaluate (10) with  $M = 1$ , (the same mode number considered in Figs. 3 and 4) since larger values of  $M$  would yield larger  $\hat{H}$ , which in turn would yield smaller growth rates due to the larger time interval between successive overreflections.

With (11) evaluated at a specified  $\text{Ri}$  for all  $\hat{k}$  and  $\hat{c}_r$  for which wave propagation between the bottom of the shear level and the ground is possible and for which the semicircle theorem of Howard (1961) is satisfied, we can determine whether the resulting maximal predicted growth rate found is a good estimate of the maximal growth rate possible for any  $\hat{k}$  and  $\hat{H}$  for the specified  $\text{Ri}$ . The agreement of these maximal growth rates will show values of  $\text{Ri}$  at which overreflection predicts growth rates accurately. Figs. 5 a-d show the resulting contours of  $\hat{k}(\hat{c}_i)_{\text{pred}}$  found using (11) and the above algorithm for  $\text{Ri} = 0.22, 0.15, 0.08$  and  $0.02$ . As noted in Part I, when  $\text{Ri} \geq 0.1164$ , the only instabilities possible are in region C of  $(\hat{k}, \hat{c}_r)$  space, where gravity waves propagate below the shear zone but are evanescent above. This is consistent with Figs. 5a and 5b, which show that for  $\text{Ri} = 0.22$  and  $0.15$ , positive growth rates predicted from overreflection only occur in region C of  $(\hat{k}, \hat{c}_r)$

space. As indicated by Figs. 5a and 5b, the region having overreflection becomes wider as  $\text{Ri}$  decreases from  $1/4$  to  $0.1164$ , but otherwise remains qualitatively similar inasmuch as it extends upwards from  $\hat{c}_r = -1$  and is confined to region C for this range of  $\text{Ri}$ . However, a qualitative change in the region having overreflection occurs when  $\text{Ri}$  decreases below  $0.116$ . When  $0.03 \leq \text{Ri} \leq 0.116$ , we found in Part I that instabilities are possible in region A of  $(\hat{k}, \hat{c}_r)$  space (where gravity waves propagate both above and below the shear zone) as well as in region C, with the largest growth rates occurring in region C, as shown in Fig. 3. This is consistent with Fig. 5c, which shows that for  $\text{Ri} = 0.08$ , the largest predicted growth rate occurs in region C, even though positive growth rates are now predicted in region A as well. Moreover, when  $0 < \text{Ri} \leq 0.03$ , instabilities are still possible in both regions A and C of  $(\hat{k}, \hat{c}_r)$  space, but now the largest growth rates occur in region A instead of region C as they do for larger Richardson numbers. This result is consistent with Fig. 5d, which shows that for  $\text{Ri} = 0.02$  the largest predicted growth rate indeed occurs in region A.

To show the quantitative accuracy with which overreflection can identify the parameters associated with maximal growth rate, we plot as a function of  $\text{Ri}$  the values of  $\hat{c}_r$  and  $\hat{k}$ , in Figs. 6 and 7, respectively, at which the maximal values of  $\hat{k}(\hat{c}_i)_{\text{pred}}$  (given by (11) and plotted in Fig. 8) are attained for the fastest growing gravity wave instability. In Figs. 9 and 10, we plot the values of  $\hat{H}$  (given by (10)) and  $R_w$  (given by (2)) which are associated with the values of  $\hat{c}_r$  and  $\hat{k}$  shown in Figs. 6 and 7. For comparison, we also plot in Figs. 6 and 7 the values of  $(\hat{c}_r)_{\text{disp}}$  and  $\hat{k}_g$  at which the maximal values of  $\hat{k}(\hat{c}_i)_{\text{disp}}$  (given by solving Eq. (27) in Part I and shown in Fig. 16 in Part I) are found, for all  $\hat{H}$  and  $\hat{k}$ , for the gravity wave instability with the largest growth rate. The values of  $\hat{H}_g$  (shown in Fig. 14 in Part I) and  $R_w$  associated with the values of  $(\hat{c}_r)_{\text{disp}}$  and  $\hat{k}_g$  shown in Figs. 6 and 7 are also plotted in Figs. 9 and 10. Figures 6-10 show that the parameters associated with the maximal value of  $\hat{k}(\hat{c}_i)_{\text{pred}}$  found from overreflection agree well with those associated with the maximal value of  $\hat{k}(\hat{c}_i)_{\text{disp}}$  found from the dispersion relation (Eq. (27) in Part I) for Richardson numbers close enough to  $1/4$  that  $R_w \leq 2$  (which is the case when  $0.18 < \text{Ri} < 0.25$ ).

To show quantitatively the closeness of the results from overreflection and those from the dispersion relation as  $\text{Ri}$  approaches  $1/4$  (indicated in Figs. 6-10), we give equations for the dependence of various parameters on  $\text{Ri}$  in this limit. These results could later be useful in parameterizing the lifetime of gravity wave instabilities and the energy that can be extracted from unstable shear flows during clear air turbulence, for example. Although we were nearly able to find the following expressions analytically using techniques also found in Lindzen and Rosenthal (1981),

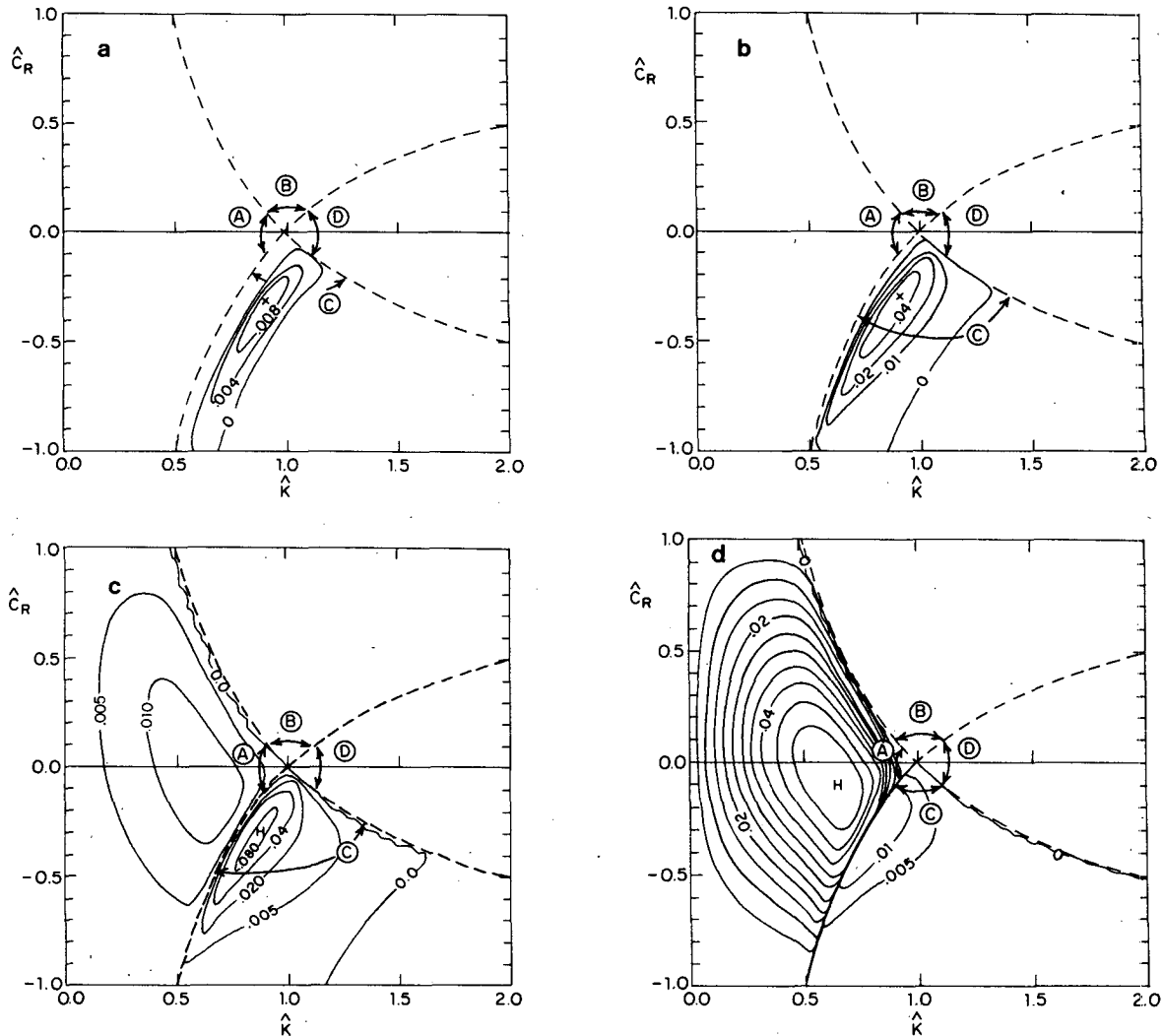


FIG. 5. Contours of predicted growth rates from overreflection found using (11) for gravity waves satisfying the quantization condition given by (10) for  $M = 1$ . Also shown are dashed lines delimiting the various regions of  $(\hat{k}, \hat{c}_r)$  space discussed in the text: (a)  $Ri = 0.22$ , (b)  $Ri = 0.15$ , (c)  $Ri = 0.08$ , (d)  $Ri = 0.02$ .

the results below are obtained from data used to prepare Fig. 5. We find that, as  $Ri$  approaches 0.25, the maximum growth rate for gravity wave instabilities occurs near  $\hat{H}_g = 3.79$  at  $\hat{k}_g = 0.904$  and may be expressed as

$$(\hat{k}\hat{c}_i)_{g,\text{disp}} = 0.292m^2 + 0.9m^4 + O(m^6), \quad (12)$$

where

$$m^2 = \frac{1}{4} - Ri. \quad (13)$$

For comparison, we note that the growth rate of the fastest growing Kelvin-Helmholtz instability at  $\hat{H}_g$  is

$$(\hat{k}\hat{c}_i)_{KH,\text{disp}} = 1.13m^2 + 5m^4 + O(m^6). \quad (14)$$

The reflection coefficient associated with gravity wave instabilities at  $\hat{H}_g$  and  $\hat{k}_g$  may be written as

$$R_w = 1 + r_w m^2 + O(m^4), \quad (15)$$

where  $r_w = 6.55$ . Numerical results using (8) also show that

$$R_{\text{eff}} = 1 + r_{\text{eff}} m^2 + O(m^4), \quad (16)$$

where  $r_{\text{eff}} = 5.87$ . At the values of  $\hat{c}_r$  and  $\hat{k}$  at which the maximal growth rate is predicted using overreflection, we find

$$R_w = 1 + r_{w,p} m^2 + O(m^4), \quad (17)$$

where  $r_{w,p} = 6.52$ , and

$$(\hat{k}\hat{c}_i)_{\text{pred}} = 0.324m^2 + O(m^4). \quad (18)$$

Since in the limit as  $Ri$  approaches  $1/4$ ,  $\hat{c}_i \ll \hat{c}_r$ , the discrepancy between  $r_w$  in (15) and  $r_{\text{eff}}$  in (16) although only 10%, is at first glance puzzling. It turns out that  $R_w$  given by (2) has a noticeably different value when evaluated with  $c_i = 0$  than when evaluated

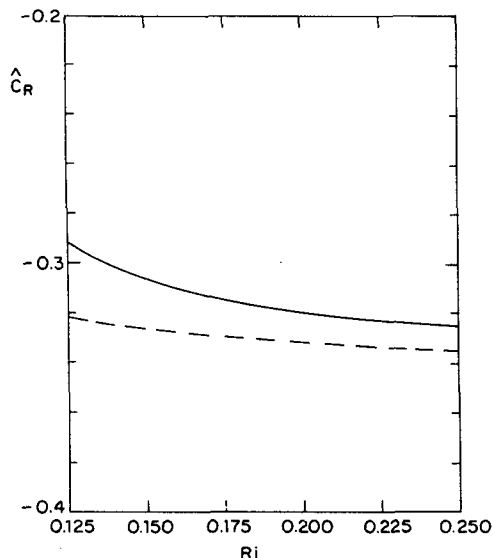


FIG. 6. The dashed line shows the phase speed for which the maximal growth rate from overreflection is predicted for gravity waves, for all  $\hat{k}$  and  $\hat{c}_r$ , at specified Richardson numbers. The solid line shows the phase speed at which the maximal dispersion growth rate is attained, for all  $\hat{k}$  and  $\hat{H}$ , for gravity waves at the specified Ri.

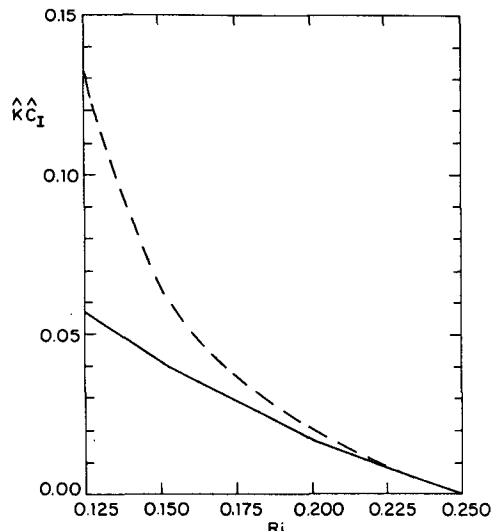


FIG. 8. The dashed line shows the maximal growth rate for gravity waves predicted from overreflection, for all  $\hat{k}$  and  $\hat{c}_r$ , as a function of Ri, the Richardson number at the critical level. The solid line shows the maximal growth rate for gravity waves according to the dispersion relation, for all  $\hat{c}_r$ ,  $\hat{k}$  and  $\hat{H}$ , as a function of Ri.

with the actual  $c_i$  attainable at any specified Ri, no matter how close Ri is to  $1/4$ . As explained by Rosenthal (1981), this accounts for  $r_w$  differing from  $r_{eff}$ , and can be exploited to yield an algorithm for predicting  $c_i$  more accurately using overreflection, even for relatively large growth rates. Nevertheless, the closeness of the coefficients in (12) and (18) already

shows the quantitative accuracy of overreflection in estimating maximal growth rates as Ri approaches  $1/4$ .

#### 4. Conclusions and suggestions

We have shown that a simple formula involving the reflection coefficient, vertical group velocity of the mode and ground to shear layer distance can give

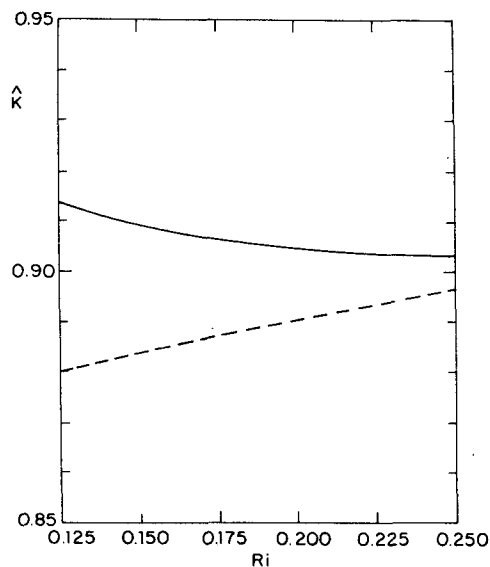


FIG. 7. As in Fig. 6, except now the horizontal wavenumber is shown for which maximal gravity wave growth rates are predicted from overreflection (dashed line) and attained according to the dispersion relation (solid line).

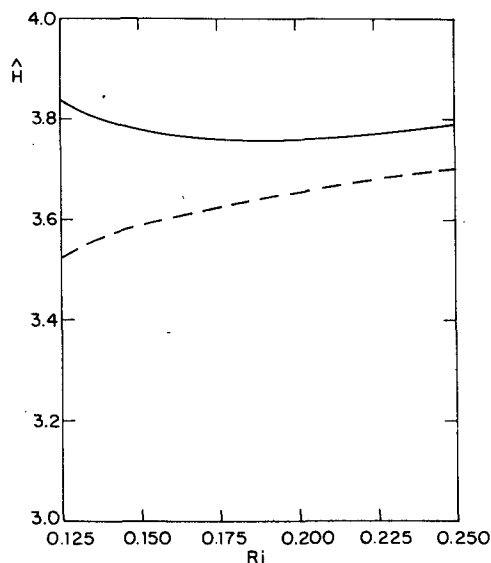


FIG. 9. As in Fig. 6, except now the values of  $\hat{H}$  occurring when the maximal gravity wave growth rates are predicted from overreflection are shown by the dashed line, and the values of  $\hat{H}_R$  at which the maximal gravity wave growth rates are attained according to the dispersion relation are shown by the solid line.

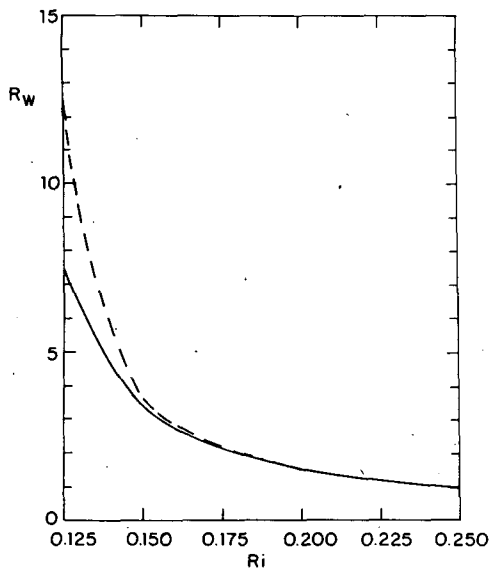


FIG. 10. As in Fig. 6, except now the values of  $R_w$  occurring when the maximal gravity wave growth rates are predicted from overreflection (dashed line) and the values of  $R_w$  occurring when the maximal gravity wave growth rates are attained according to the dispersion relation (solid line) are shown.

quantitative estimates of the growth rate that are especially accurate for small growth rates. Moreover, considering overreflection enables the physical parameters associated with the fastest growing gravity wave to be accurately estimated. The relative constancy of the variables  $\hat{c}_r$ ,  $\hat{k}$  and  $\hat{H}$  at which maximal growth rates for gravity waves are attained for different Richardson numbers (as shown in Figs. 6, 7 and 9) has already been discussed in terms of the constraints imposed by quantization [see Lindzen and Rosenthal (1976) and Part I], and will not be explained in detail again here. Physically, this constancy is important because it means that if a shear profile is smoothed to make the Richardson number increase towards  $\frac{1}{4}$ , a gravity wave at a fixed wavelength and phase speed can perhaps grow to a large finite amplitude. This behavior contrasts with that for Kelvin-Helmholtz instabilities, for which the most unstable wavelength increases as the shear layer is broadened. This implies that a Kelvin-Helmholtz instability at a fixed wavelength and phase speed would only grow for a limited range of Richardson numbers.

At first glance, the above statement that a shear profile would be smoothed to make the Richardson number increase towards  $\frac{1}{4}$  appears to be an incorrect assumption, since McIntyre and Weissman (1978) apparently show that a Helmholtz velocity profile with zero Richardson number would not be smoothed to make the Richardson number increase towards  $\frac{1}{4}$  as waves radiate energy away from the shear layer. For such a profile, the mean flow changes only at the

locations where the wave field is changing as the radiating wave from the shear layer reaches these locations; the mean flow does not change in the neighborhood of the shear layer. However, Davis and Peltier (1977) show, by considering the Reynolds stress, that a velocity profile with nonzero Richardson number in the shear zone would indeed be smoothed in the manner assumed above. Whether the smoothing is adequate remains to be determined. Observationally, gravity waves are prominent [Gossard *et al.* (1971); Reed and Hardy (1972); Ottersten *et al.* (1973)].

Whereas finding the largest growth rate using the dispersion relation [Eq. (27) in Part I] involves a maximization over the three variables  $\hat{c}_r$ ,  $\hat{k}$  and  $\hat{H}$  of a function  $\hat{k}(\hat{c}_i)_{\text{disp}}$ , found by solving a transcendental equation in the two variables  $\hat{c}_r$  and  $\hat{c}_i$ , predicting the largest growth rate using overreflection involves a maximization over the two variables  $\hat{k}$  and  $\hat{c}_r$  of a function,  $\hat{k}(\hat{c}_i)_{\text{pred}}$  which can be explicitly calculated using (2) and the quantization condition given by (10). It turns out that considering overreflection and recognizing an appropriate quantization condition can also reduce the number of variables that have to be searched over in order to identify the maximal growth rates associated with gravity wave instabilities in stratified shear flows that have more than one critical level.

The question of why overreflection estimates differ from explicit instability results at "large" values of  $c_i$  will be dealt with in a separate paper. Briefly, as McIntyre and Weissman (1978) showed, overreflection takes time to develop and when this time exceeds  $2Hv_g^{-1}$  then we cannot expect results for  $c_i = 0$  to be correct. It will be shown that when this effect is included, excellent agreement between overreflection and instability results is obtained. Also, for large  $c_i$  it proves helpful to calculate  $v_g$  more accurately. Ultimately the important point is that we can identify a class of instabilities with the interaction of internal gravity waves with a mean flow.

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