# Charney's Problem for Baroclinic Instability Applied to Barotropic Instability

R. S. LINDZEN
Center for Earth and Planetary Physics, Harvard University, Cambridge, MA 02138

A. J. ROSENTHAL
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

B. FARRELL
Center for Earth and Planetary Physics, Harvard University, Cambridge, MA 02138

(Manuscript received 16 September 1982, in final form 8 November 1982)

ABSTRACT

It is suggested that barotropically unstable easterly jets may be approximated by broken line profiles. It is then possible to use solutions to the Charney problem for baroclinic instability in order to solve the barotropic problem. Results are presented for easterly jets with an arbitrary degree of asymmetry.

## 1. Introduction

The fundamental instability mechanisms for large scale atmospheric flows are baroclinic instability and barotropic instability. Baroclinic instability is of primary importance at middle and high latitudes, while barotropic instability appears to be of central importance in the tropics. We will discuss this matter later.

As noted by Lindzen and Tung (1978), both instabilities have fundamentally the same origin: the conservation of potential vorticity. Nonetheless, baroclinic instability has received the most attention. Several analytically treatable prototypes for baroclinic instability are common in the literature: the 2-layer model (Phillips, 1954), the Eady model (Eady, 1949) and the Charney problem (Charney, 1947). The first two are extremely popular because of their almost trivial mathematical solution. The solution of the Charney problem is rather difficult; however, a convenient analytic treatment has recently been developed by Lindzen and Rosenthal (1981). Oddly enough, no comparable prototypes have been developed for the barotropic instability problem. Most treatments involve the numerical solution for the stability properties of tanh-type shear layers (Dickinson and Clare, 1973), sech$^2$-type jets (Kuo, 1973) or other more realistic profiles (Kuo, 1978). The purpose of the present paper is to show that a very reasonable prototype for barotropic instability in the atmosphere is mathematically equivalent to the Charney problem with, however, slightly modified boundary conditions (except in one important case where even the boundary condition is the same). The treatment of Lindzen and Rosenthal (1981) is then applied to a variety of cases.

In this paper we will restrict ourselves to a barotropic, non-divergent atmosphere on a $\beta$-plane. The equation governing linearized perturbations on a basic zonal flow $\bar{u}(y)$ is

$$\psi_{yy} + \left[\frac{\beta - \bar{u}_{yy}}{\bar{u} - c} - k^2\right]\psi = 0, \qquad (1)$$

where solutions of the form $\Psi = \psi(y)\exp[ik(x - ct)]$ have been assumed for the streamfunction and $f$ = Coriolis parameter; $\beta = df/dy$; $x$ = eastward distance; and $y$ = northward distance.

Fig. 1 shows a schematic distribution of 500 mb zonal wind $\bar{u}$ as a function of latitude. Also shown is $\beta - \bar{u}_{yy}$ ($\partial/\partial y = a^{-1}\partial/\partial\phi$, where $a$ is the radius of the earth and $\phi$ the latitude). Recall that a necessary condition for barotropic instability is that $\beta - \bar{u}_{yy}$ change sign (Kuo, 1949). Note that $\beta$ is positive as is $\bar{u}_{yy}$ due to the westerly jet in middle latitudes. Only the cessation of positive $\bar{u}_y$ in the tropics is associated with negative $\bar{u}_{yy}$ and a change in the sign of $\beta - \bar{u}_{yy}$. In general, only the behavior of $\bar{u}$ in the neighborhood of the region of negative $\beta - \bar{u}_{yy}$ will prove important for barotropic instabilities. Anticipating a value of $c$ corresponding to $\bar{u}$ just poleward of the region of negative $\beta - \bar{u}_{yy}$, we note that for relevant values of $k$, the bracketed quantity in (1) will have a turning point well before the westerly maximum is reached. As noted by Lindzen and Rosenthal (1981), the behavior of $\bar{u}$ beyond this point is largely irrelevant to the instability problem. Our prototype problem will

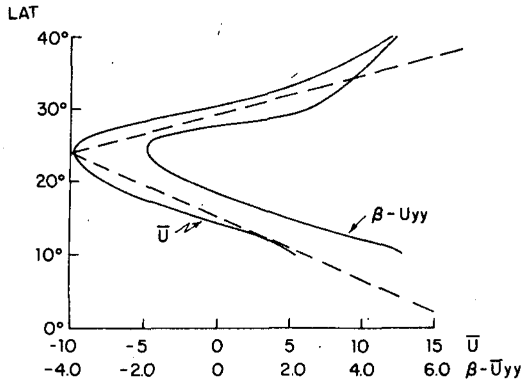


FIG. 1. Properties of a zonal flow at 500 mb that is characteristic of the troposphere in July. The solid lines show the observed zonal wind $\bar{u}$ and the potential vorticity gradient $\beta - u_{yy}$ versus latitude. The dashed curve shows an approximation to $\bar{u}$ used in the calculations.

consist in approximating $\bar{u}$ in Fig. 1 with a broken line profile. The effects of the resulting $\delta$-function contribution to $\bar{u}_{yy}$ will be discussed later. Before proceeding to consider the general broken line profile, we will consider two particularly useful special cases.

## 2. Symmetric easterly jet

In this section we will consider the broken line representation of a symmetric easterly jet (viz. Fig. 2). For convenience we will take the origin of our $\beta$-plane to coincide with the jet vertex. Thus,

$$\left.\begin{aligned} u &= u_0 + my, \quad y > 0 \\ &= u_0 - my, \quad y < 0 \end{aligned}\right\}.$$

Our boundary conditions are that $\psi$ remain bounded as $|y| \to \infty$. For $y > 0$, (1) becomes

$$\psi_{yy} + \left[\frac{\beta}{u_0 + my - c} - k^2\right]\psi = 0. \tag{2}$$

By symmetry $\psi(y) = \psi(-y)$.

At $y = 0$ we require continuity of $\psi$. We obtain a condition on $\psi_y$ by integrating (2) across $y = 0$:

$$\psi_y \Big|_-^+ - \frac{\bar{u}_y \Big|_-^+}{u_0 - c}\psi = 0,$$

$$\psi_y^+ - \frac{m}{u_0 - c}\psi = \psi_y^- + \frac{m}{u_0 - c}\psi, \quad \text{at} \quad y = 0. \tag{3}$$

By symmetry $\psi_y^+ = -\psi_y^-$, and (3) becomes

$$\psi_y^+(0) - \frac{m}{u_0 - c}\psi^+(0) = 0. \tag{4}$$

Eq. (2) together with the boundary condition (4) and the requirement of boundedness as $y \to \infty$, are precisely the equations of the Charney problem for baroclinic instability in a Boussinesq fluid. As noted by Lindzen and Farrell (1980a), the explicit dependence of this problem on the value of the shear, $m$, or on $u_0$, can be removed by means of scaling and coordinate definitions.

For the present problem we let

$$\zeta = \frac{\beta}{m}\left(y - \frac{c - u_0}{m}\right), \tag{5a}$$

$$\alpha^2 = \frac{m^2}{\beta^2}k^2, \tag{5b}$$

$$\tilde{c} = \frac{\beta}{m}\frac{c - u_0}{m}. \tag{5c}$$

Eq. (2) becomes

$$\psi_{\zeta\zeta} + \left[\frac{1}{\zeta} - \alpha^2\right]\psi = 0, \tag{6}$$

while the boundary condition (4) becomes

$$\psi_\zeta + \frac{1}{\tilde{c}}\psi = 0 \quad \text{at} \quad \zeta = -\tilde{c}. \tag{7}$$

Eqs. (6) and (7) have already been solved in connection with baroclinic instability (Lindzen and Farrell, 1980a; Lindzen and Rosenthal, 1981). The results for $\alpha\tilde{c}_i$ versus $\alpha$ and $\tilde{c}_r$ versus $\alpha$ are shown in Fig. 3. Numerical values for any particular choice of $m$ can be obtained by means of relations (5).

As already noted (Burger, 1962; Lindzen *et al.*, 1980) the curves of $\alpha\tilde{c}_i$ versus $\alpha$ and $\tilde{c}_r$ versus $\alpha$ for the Charney problem in a Boussinesq fluid have an infinite number of zeroes with a collection point at $\alpha = 0$. This somewhat peculiar feature is eliminated by either the inclusion of non-Boussinesq terms in the baroclinic case or divergence in the barotropic case. Moreover, as $\alpha \to 0$ in the barotropic case, the meridional scale gets large enough for our prototypical basic state to become questionable. Finally, growth rates diminish as $\alpha \to 0$. As a result, we will confine our attention to the two modes associated with the largest values of $\alpha$ ($1/2 \lesssim \alpha \lesssim \infty$, $1/4 \lesssim \alpha \lesssim 1/2$).

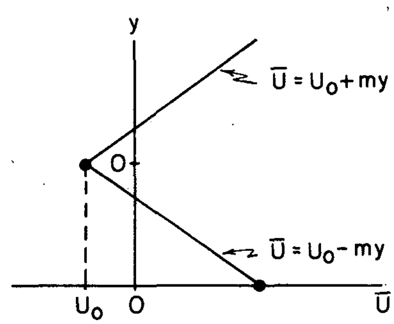


FIG. 2. Zonal wind $\bar{u}$ versus latitude for the broken-line representation of a symmetric easterly jet (solid curve) and a semi-jet (dashed curve).

## 3. Easterly semi-jet

Another special case which is only slightly different from the Charney problem is the semi-jet also shown in Fig. 2. In this case

$$\left.\begin{aligned}\bar{u} &= u_0 + my, \quad & y > 0\\ \bar{u} &= u_0, & y < 0\end{aligned}\right\}.$$

Eq. (2) remains appropriate for $y > 0$. For $y < 0$ we have

$$\psi_{yy} + \left[\frac{\beta}{u_0 - c} - k^2\right]\psi = 0. \quad (8)$$

The solution for $y < 0$ is simply

$$\psi^- = Ae^{\mu y}, \quad (9)$$

where $\mu$ is the root of

$$\mu = \left[k^2 - \frac{\beta}{u_0 - c}\right]^{1/2}$$

with a positive real part. Let us refer to the solution for $y > 0$ as $\psi^+$. At $y = 0$ we require continuity of $\psi$: i.e.,

$$\psi^+ = \psi^- \quad \text{at} \quad y = 0.$$

Integrating (2) across $y = 0$ we also obtain

$$\psi_y^+ - \psi_y^- = \frac{m\psi^+}{u_0 - c}, \quad (10a)$$

or using (9)

$$\psi_y^+ = \left[\frac{m}{u_0 - c} + \mu\right]\psi^+ \quad \text{at} \quad y = 0. \quad (10b)$$

Eq. (10b) differs from (4) only in the presence of $\mu$ in the brackets.

Using the relations (5), we again have (6) for $\psi^+$, but instead of (7) we now have

$$\psi_\zeta^+ + \left[\frac{1}{\tilde{c}} - \left(\alpha^2 + \frac{1}{\tilde{c}}\right)^{1/2}\right]\psi^+ = 0 \quad (11)$$

at $\zeta = -\tilde{c}$. The solutions to (6) obtained by Lindzen and Rosenthal (1981) are still appropriate. The modified boundary condition is handled with no difficulty and the results are shown in Fig. 4.

The results are similar to those in Fig. 3—including the position of the "neutral" points.[1] However, there are two notable differences. Maximum values of $\tilde{c}_r$ and of growth rates $\alpha\tilde{c}_i$ are about half of what is obtained for the symmetric "full" jet. Viewed in terms of wave-overreflection (Lindzen *et al.*, 1980), the wave at the vertex is bounded by two overreflecting regions in the case of the symmetric jet, and by one overreflecting region and a "turning point" in the case of the semi-jet. This striking difference in growth rates between symmetric jets and semi-jets will be shown, in a companion paper, to play a significant role in explaining the onset of monsoon depressions.

As noted in connection with the symmetric jet, the behavior at very small values of $\alpha$ is unlikely to be relevant.

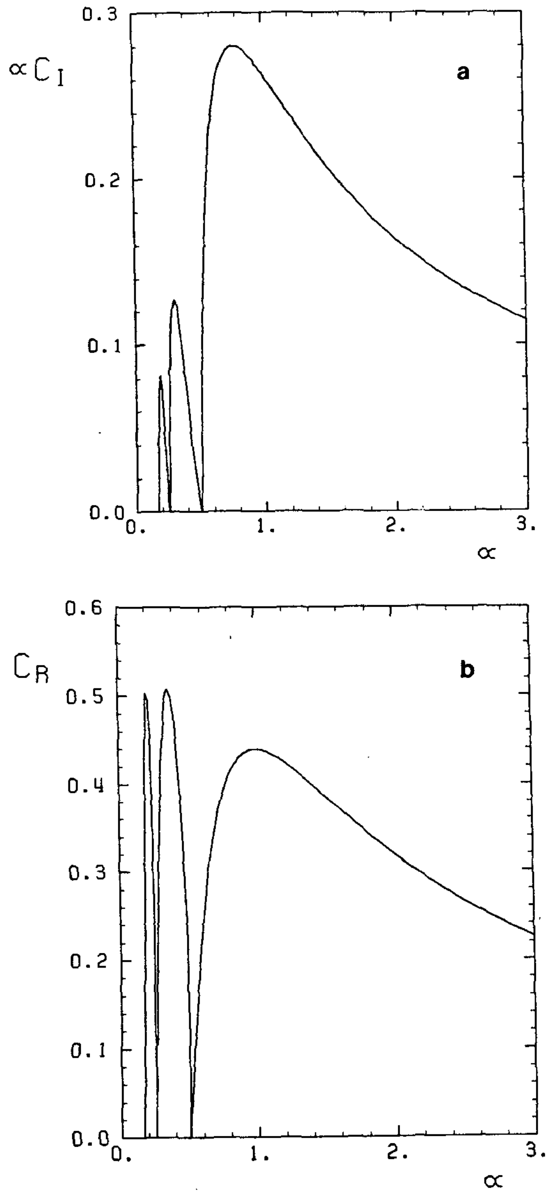


FIG. 3. Nondimensional growth rate $\alpha\tilde{c}_i$ and phase speed $\tilde{c}_r$ versus horizontal wavenumber $\alpha$ for the broken-line representation of a symmetric easterly jet illustrated by the solid line in Fig. 2.

## 4. The asymmetric jet, I

The extension of the results in Section 2 to an asymmetric jet is straightforward. For $\bar{u}$ we now take

[1] As noted by Burger (1966), these points are not strictly neutral; they are associated with linear rather than exponential growth in time.

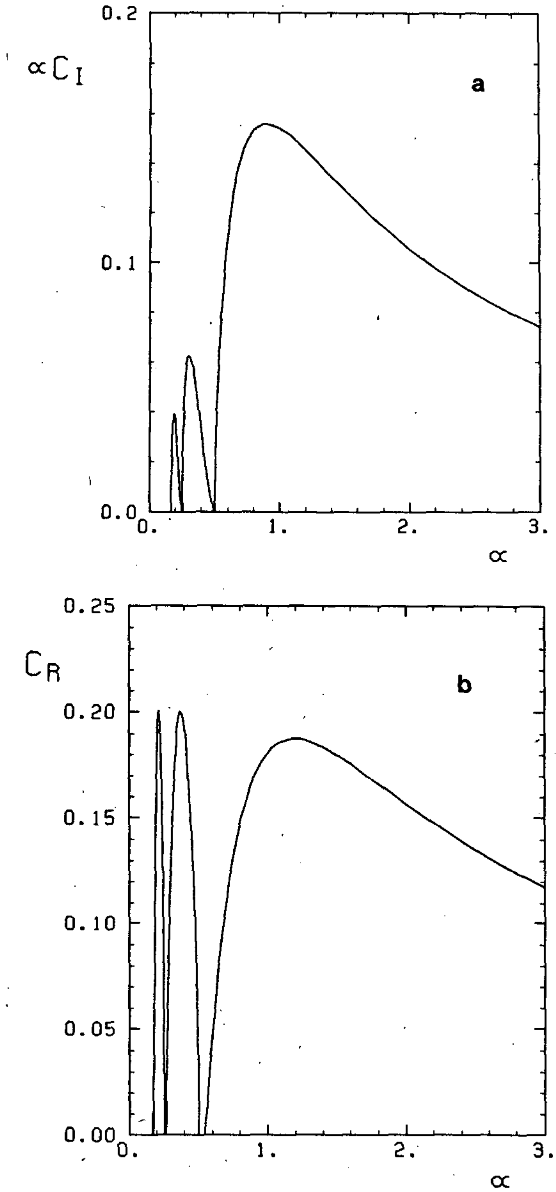


FIG. 4. As in Fig. 3 except for the semi-jet illustrated by the dashed curve in Fig. 2.

$$\left.\begin{aligned} \bar{u} &= u_0 + my, \quad y \geq 0 \\ \bar{u} &= u_0 - ny, \quad y \leq 0 \end{aligned}\right\}.$$

We retain the scaling relations given by (5) for $y > 0$. For $y < 0$, we retain (5b) and (5c), but replace (5a) with

$$\eta = \frac{\beta}{m}\left(-y - \frac{c - u_0}{n}\right). \tag{12}$$

Then (2) becomes, for $y < 0$,

$$\psi_{\eta\eta} + \left[\frac{mn^{-1}}{\eta} - \alpha^2\right]\psi = 0. \tag{13}$$

Both (6) and (13) have solutions given by Lindzen and Rosenthal (1981). The stability (and dispersion) relations are again obtained from the matching conditions at $y = 0$. Let $\psi(\zeta) = \psi^+(\zeta)$ for $y > 0$, and let $\psi(\eta) = \psi^-(\eta)$ for $y < 0$. Here $y = 0$ corresponds to $\zeta = -\tilde{c}$ and to $\eta = -mn^{-1}\tilde{c}$. Similarly

$$\psi_y = \beta m^{-1}\psi_\zeta^+ = -\beta m^{-1}\psi_\eta^-.$$

Continuity of $\psi$ at $y = 0$ yields

$$\psi^+(-\tilde{c}) = \psi^-(-mn^{-1}\tilde{c}), \tag{14a}$$

while (3) becomes

$$\psi_\zeta^+(-\tilde{c}) + \tilde{c}^{-1}\psi^+(-\tilde{c}) = -\psi_\eta^-(-mn^{-1}\tilde{c}) - nm^{-1}\tilde{c}^{-1}\psi^-(-mn^{-1}\tilde{c}). \tag{14b}$$

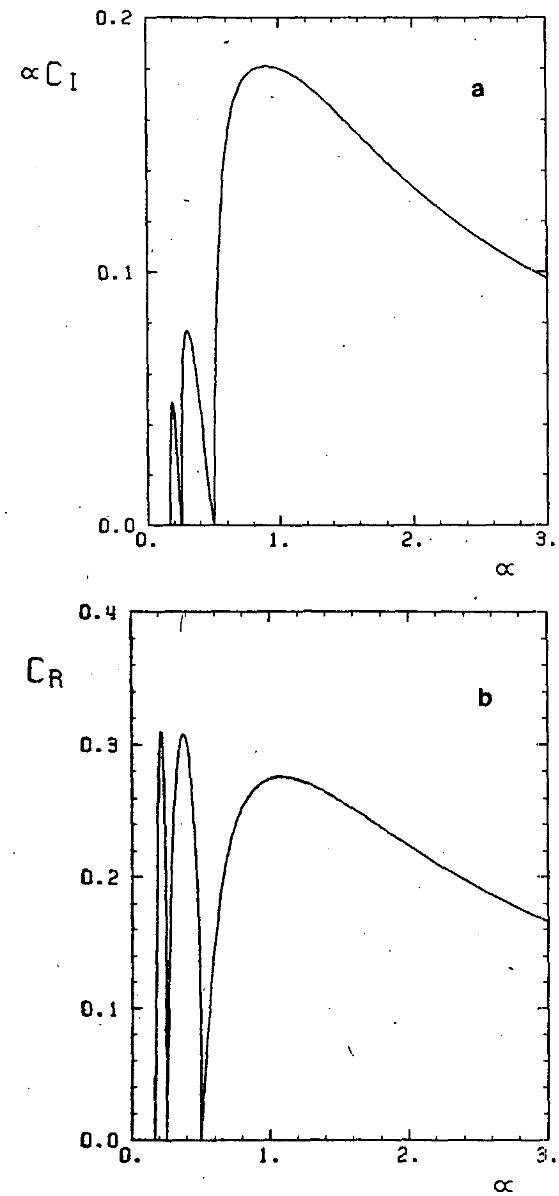


FIG. 5. Stability curves for the asymmetric jet described in Section 4 having slope ratio $n/m = 0.5$.

The implementation of (14a) and (14b) is somewhat more tedious than it is for the simpler cases described in Sections 2 and 3. However, we may restrict ourselves to $nm^{-1} < 1$ because of the symmetry of the present problem about $y = 0$.

Figures 5 and 6 show the stability curves for $nm^{-1} = 0.5$ and $nm^{-1} = 0.25$ respectively. Note that the maximum value of $\alpha\tilde{c}_i$ is diminishing for diminishing $nm^{-1}$—consistent with the results in Sections 2 and 3.

## 5. The asymmetric jet, II

The results in Sections 2 and 4 involve instabilities with two critical latitudes ($\bar{u} = c_r$). It may be anticipated, however, that the wind south of $y = 0$ may not continue increasing far enough to allow a critical latitude. The situation contemplated is schematically illustrated in Fig. 7. Consider, for example, an easterly jet where the shear just south of $y = 0$ is $-n$, but where the shear disappears before a critical value of $\bar{u}$ is reached. Then for $y < 0$ the bracketed quantity in (1); i.e.,

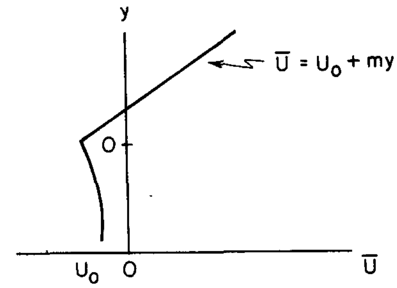


FIG. 7. Zonal wind $\bar{u}$ versus latitude for an asymmetric jet described in Section 5 having only one critical latitude.

$$-\mu^2 = \left[\frac{\beta}{\bar{u} - c} - k^2\right] \tag{15}$$

will always be negative and will not approach a singularity. Using a WKB approach we then get

$$\psi^-(y) \approx A\mu^{-1/2} \exp\left[\int_0^{-y} \mu(y')dy'\right], \tag{16}$$

and

$$\psi_y^-(y) \approx \mu\psi^-.$$

The remaining analysis is almost identical to that of the semi-jet of Section 3 except that (10b) is replaced by

$$\psi_y^+ = \left\{\frac{m + n}{u_0 - c} + \mu\right\}\psi^+ \quad \text{at} \quad y = 0, \tag{17}$$

or in terms of the scaled variables, (11) becomes

$$\psi_\zeta^+ + \left\{\frac{1 + nm^{-1}}{\tilde{c}} - \left[\alpha^2 + \frac{1}{\tilde{c}}\right]^{1/2}\right\}\psi^+ = 0 \tag{18}$$

at $\zeta = -\tilde{c}$. Stability results, based on (18), for $nm^{-1} = 1$ are shown in Fig. 8. The results are similar to those for the semi-jet, except that $\tilde{c}_r$ no longer approaches zero for isolated values of $\alpha$. Zeroes for $\tilde{c}_r$ would, of course, be inconsistent with the assumption of no critical latitudes for $y < 0$.

In practice we would first solve the problem of an easterly jet with the approach of Section 4. If the results predicted an unrealizable critical latitude for $y < 0$, we would revert to the analysis of this section.

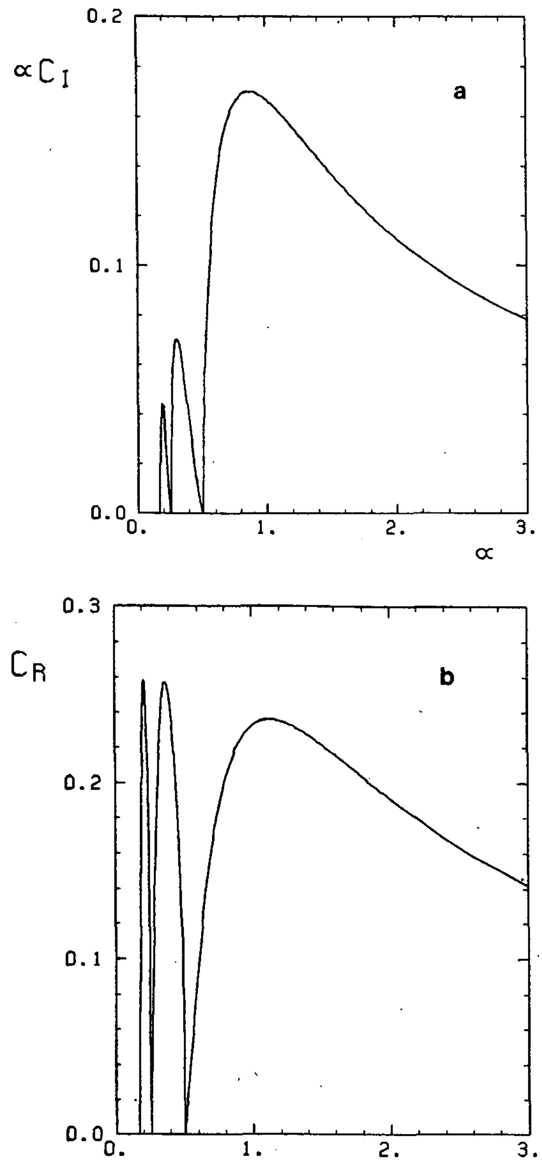


FIG. 6. As in Fig. 5 except for the asymmetric jet with $n/m = 0.25$.

## 6. Concluding remarks

We have shown that a reasonable prototype for barotropically unstable easterly jets (or semi-jets) can be solved using readily obtained solutions for Charney's baroclinic instability problem. As noted by Farrell (1982), the advantage of this approach is that it

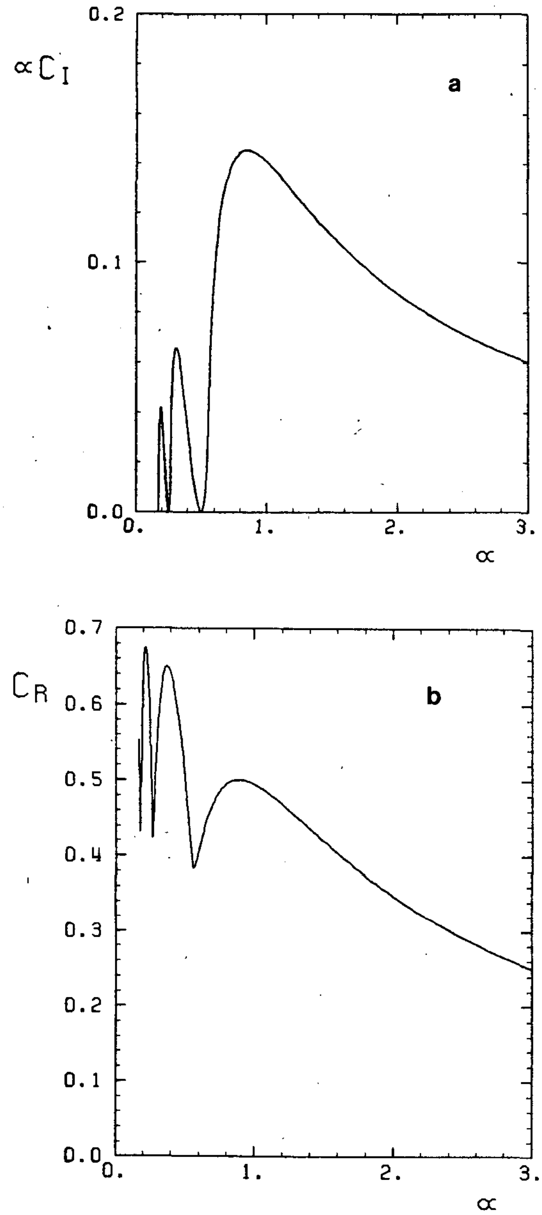


FIG. 8. Stability curves for the asymmetric jet having one critical level described in Section 5 and illustrated schematically in Fig. 7 having slope ratio $n/m = 1$.

provides analytic expressions for instability and dispersion valid for complex $c$ and $k$. This, in turn, allows for the convenient calculation of the growth of localized as well as normal mode perturbations. This is especially important to the study of monsoon depressions over India given in a companion paper (Lindzen *et al.*, 1983). The present paper already shows that the growth rate of perturbations on a symmetric jet is about double that on a semi-jet with a continuous variation for intermediate degrees of asymmetry. Finally, there appear to be counterparts in barotropic instability to Burger–Green modes as well as to Charney modes. A reasonable question arises as to how dependent our results are on the replacement of the smooth jets in Fig. 1 with the sharp corner of the broken line profile. It is clear from Lindzen *et al.*, 1980 that the corner can be slightly smoothed with no significant change in the results. Moreover, Lindzen and Farrell (1980b) have shown that as the corner is smoothed, maximum growth rates decrease very slowly until $\beta - \bar{u}_{yy}$ at the jet is brought nearly to zero (at which point neutrality is obtained). However, one may expect the separation between Burger–Green modes and Charney modes to become less sharp. These matters remain to be investigated more carefully.

*Acknowledgments.* The authors wish to acknowledge the support of the National Science Foundation under Grant ATM-78-23330 at Harvard and Grant ATM-802-3523 at MIT, and the National Aeronautics and Space Administration under Grant NGL-22-007-228.

REFERENCES

Burger, A. P., 1962: On the non-existence of critical wavelengths in a continuous baroclinic stability problem. *J. Atmos. Sci.*, **19**, 31–38.

——, 1966: Instability associated with the continuous spectrum in a baroclinic flow. *J. Atmos. Sci.*, **23**, 272–277.

Charney, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, **4**, 135–162.

Dickinson, R. E., and F. J. Clare, 1973: Numerical study of the unstable modes of a hyperbolic-tangent barotropic shear flow. *J. Atmos. Sci.*, **30**, 1035–1049.

Eady, E. J., 1949: Long waves and cyclone waves, *Tellus*, **1**, 33–52.

Farrell, B. F., 1982: Pulse asymptotics of the Charney baroclinic instability problem. *J. Atmos. Sci.*, **39**, 507–517.

Kuo, H. L., 1949: Dynamic instability of two-dimensional nondivergent flow in a barotropic atmosphere. *J. Meteor.*, **6**, 105–122.

——, 1973: Dynamics of quasi-geostrophic flows and instability theory. *Advances in Applied Mechanics*, Vol. 13, Academic Press, 247–330.

——, 1978: A two-layer model study of the combined barotropic and baroclinic instability in the tropics. *J. Atmos. Sci.*, **35**, 1840–1860.

Lindzen, R. S., and K. K. Tung, 1978: Wave over-reflection and shear instability. *J. Atmos. Sci.*, **35**, 1626–1632.

——, and B. Farrell, 1980a: A simple approximate result for the maximum growth rate of baroclinic instabilities. *J. Atmos. Sci.*, **37**, 1648–1654.

——, and ——, 1980b: The role of polar regions in global climate, and a new parameterization of global heat transport. *Mon. Wea. Rev.*, **108**, 2064–2079.

——, and A. J. Rosenthal, 1981: A WKB asymptotic analysis of baroclinic instability. *J. Atmos. Sci.*, **38**, 619–629.

——, B. Farrell and K. K. Tung, 1980: The concept of wave overreflection and its application to baroclinic instability. *J. Atmos. Sci.*, **37**, 44–63.

——, —— and A. J. Rosenthal, 1983: Absolute barotropic instability and monsoon depressions. *J. Atmos. Sci.*, (in press).

Phillips, N. A., 1954: Energy transformations and meridional circulations associated with simple baroclinic waves in a two-level, quasi-geostrophic model. Tellus, **6**, 273–286.