NOTE AND CORRESPONDENCE

Comments on "On the Shear Instability without Over-reflection," by Masaaki Takahashi (J. Met. Soc. Japan, Vol. 64, No. 6, pp 793-804)

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Takahashi (1986), includes in his paper calculated stability curves (*i.e.*, curves of the imaginary and real parts of the phase speed $-c_i$ and c_r $-$ of the unstable perturbation as functions of the zonal wavenumber, k) for three shear problems. We have no reason to question most of the explicit results. However, on several counts, this paper is disturbingly misleading, and on its main point (namely, that the instabilities described do not involve wave over-reflection), the paper is simply incorrect.

In section 1 of Takahashi (1986) (to be referred to as T), the author describes 4 conditions which he claims that we asserted were necessary conditions for over-reflection. This is not quite true. What we did (Lindzen and Tung, 1978) was to note the wave geometry associated with over-reflection in barotropic shear flow, and suggest that this geometry might have broader significance. In this connection it should be noted that for the particular problem we looked at, the exponential region existed only between the critical level and the over-reflecting wave region (not on both sides as suggested by condition 2 in T) and it was, moreover, stated that condition 4 in T was not essential. Far more important is the fact that the conditions suggested by Lindzen and Tung (1978) were subsequently studied at considerable length. Lindzen and Barker (1985) repeat the conditions suggested by Lindzen and Tung (1978) in order to correct, generalize, and physically interpret them. T does refer to Lindzen and Barker (1985) —but for the most part he ignores its contents. We will review the revised conditions arrived at by Lindzen and Barker (1985) later in this note, since they also pertain to the relation between over-relfection and instability. Less important, but still disturbing, is T's acknowledgement of correspondence with one of us (R.S. Lindzen). Generally, such an acknowledgement is taken to mean that exchanges actually took place, and that there was some measure of mutual understanding. In the present case, Takahashi sent one of us (R.S. Lindzen) a preprint of T and a request for comments. In response, a letter was sent to Takahashi expressing disagreement with his conclusion; also, a reprint of Lindzen and Rosenthal (1983) was sent to Takahashi. This paper, which Takahashi does not refer to, was Part III of a three part paper, all of which appeared in the same issue of J. Atmos. Sci. Takahashi did refer the first two parts (Rosenthal and Lindzen, 1983a, and 1983b) so it is hard to imagine that he was unaware of the third part. It was in Part III that it was shown that instabilities of exactly the same type as those discussed in *T* were, in fact, associated with the over-reflection of vorticity waves. We have no way of being sure that Takahashi actually received the above mentioned letter since he neither answered the letter nor took account of it in the final version of *T*.

In order to clarify matters for the readers, we will briefly review our current understanding of the relation between over-reflection, shear instability, and wave geometry. Details may be found in Lindzen and Barker (1985) (see also Lindzen, 1987; it should be added that our current understanding may be flawed, but we prefer that criticism be addressed to our current views rather than to older – and partly rejected – suggestions). The following conditions appear to us (on the basis of many explicit calculations) to be necessary for both wave over-reflection and instability in plane parallel shear flows:

1) There must be some surface in the fluid where there is shear, and where the disturbance is travelling with the mean flow. This feature is traditionally addressed by means of the famous semicircle theorems (Howard, 1961, Pedlosky, 1964). In general, this will be a critical level, though the results of Boyd (1983) and Pedlosky (1964) suggest that in the presence of β, this may not always be precisely the case. For a disturbance travelling with the flow, shear can amplify the disturbance (Orr, 1907, Farrell, 1981).

2) There must be some region in the flow where the disturbance can propagate as a wave in the direction normal to the critical surface. This condition is obviously necessary in order to even speak of wave over-reflection. However, as we shall soon note, it may also be essential for shear instability.

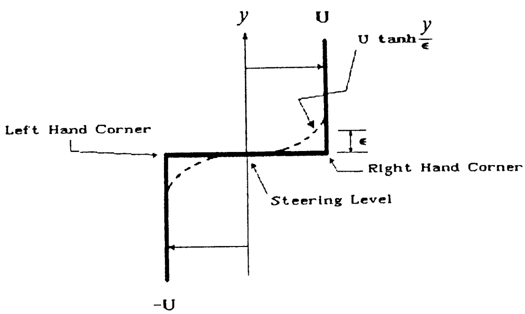
3) There must be some way for the wave described in item 2 to reach the surface where the disturbance is travelling with the mean flow. If the basic state is smooth (*i.e.*, the flow and the shear have no discontinuities – as is always the case in real situations), if the fluid is inviscid, and if this surface is a critical level (or surface) within the region of propagation, then we will usually have a problem meeting this condition. As a rule, the wave's group velocity will approach zero so rapidly that the wave will never reach the critical surface. A number of ways of getting around this difficulty are described in Lindzen and Barker (1985). The most common approach to this difficulty is to have a region of exponential (non propagating) behavior between the critical surface and the wave region[1]. This is often enough for the wave perturbation to reach the critical level. However, for the critical level to produce an amplifying disturbance, condition 4 also appears necessary.

4) There must be a sink for the wave flux on the side of the critical surface opposite to the wave region. Without such a sink, the wave will not "genuinely" tunnel across the critical surface (We will expand on the word "genuinely" soon). Such a sink can be provided by a second wave region. (This is condition 3 in *T*). Given that this wave region's primary role is to provide a sink for wave flux, there is no reason to bound this wave region (counter to condition 4 in *T*). Indeed, Lindzen and Barker (1985) show that this second wave region can be replaced by a region of enhanced linear damping. Similarly, Lindzen and Rambaldi (1986) show that in a viscous fluid, wall boundary layers can serve as wave flux sinks. The need for this sink warrants a bit more discussion as does the related matter of "genuine" tunnelling. Clearly, even in the absence of the sink, the wave perturbation will not go to zero abruptly as one enters the exponential region; rather, the perturbation will penetrate this region, decaying exponentially. Thus, the perturbation might, in some sense, be said to have reached the critical level even without the presence of a sink. In this case, however, the wave flux is zero even though the wave amplitude is not. This is at least one reason for saying that we do not have "genuine" tunnelling in this case. More important, is the

[1] Lindzen and Rambaldi (1986) showed that for a viscous fluid the disturbance can reach the critical surface by means of diffusion.

fact that the Orr mechanism does not amplify all disturbances; it only amplifies disturbances with phases tilting in a direction opposite to that of the shear[2]. We have found that this is not the case for perturbations at the critical level when we do not have over-reflection, but that it is the case when we do have over-reflection (see Lindzen and Rambaldi, 1986, for an explicit example). This strongly supports the suggestion that it is the Orr mechanism that gives rise to amplification and over-reflection. The process is referred to as "stimulated emission" in Lindzen (1987).

The above appear to be the correct necessary conditions for over-reflection. In order for an over-reflected wave to give rise to an unstable mode, we also require that the wave region in item 2 be bounded (by a wall, a turning point, or even a change in the index of refraction sufficient to give significant partial reflection) and that its depth be such that we can have an infinite sequence of over-reflections and reinforcing reflections as in Lindzen and Rosenthal (1976). However, it is the claim of T that not all shear instabilities involve over-reflection. For reasons we will discuss at the end of this note, we don't believe this to be the case; though our arguments are not rigorous. However, it is relatively easy to show that it is at least not the case for the examples presented in *T*. All of *T*'s



Tanh Profile for U and Helmholtz Idealization

Fig. 1. The Helmholtz velocity profile. Also shown (by dashed lines) is a continuous profile to which the Helmholtz profile may be regarded as an approximation.

[2] The Orr mechanism (due to Orr, 1907) is discussed further in the final paragraph of these remarks. References are given there to articles offering detailed descriptions of the mechanism.

examples involve basic flows with discontinuities in shear and/or velocity. *T* solves the equations away from the discontinuities and matches the solutions across the discontinuities. Although all of *T*'s examples have been designed to have wave regions, this is not essential for his argument. The simplest relevant example is the instability of an unstratified Helmholtz profile in a non-rotating fluid (Fig. 1 here; Fig. 2 in *T* without β, and Fig. 12 in *T* with N=0). The solution to this classic problem has been known for over a century (Helmholtz, 1868, Drazin and Reid, 1981). Instabilities exist with $c_i = U$; the associated eigenfunctions are of the form $e^{-k|z|}$. Superficially, we appear to have instability without any wave regions at all!

The above result is, however, untrue as concerns the non-existence of wave regions. The matching conditions are a way of avoiding the need to solve the equation at the discontinuities, but it is still essential for our purposes to see what the equation looks like at these points. For the simple Helmholtz problem, the correct equation is

$$\frac{d^2\phi}{dy^2} + \left(\frac{-U_{yy}}{U-c} - k^2\right)\phi = 0 \qquad (1)$$

For broken line profiles we can ignore the U_{yy} term away from discontinuities; however, it is clear that this term behaves very dramatically at a discontinuity. This fact does not change simply because we can bury this behavior in the matching conditions. From Fig. 1, it is clear that U_{yy} has a negative δ-function contribution at the right hand corner of *U*, and a positive δ-function contribution at the left hand corner. Moreover, the instabilities are travelling with the flow at *U=0*, which lies between the two corners and is associated with infinite shear. Because *U*-c changes sign between the two corners in the *U*-profile, each of the δ-function regions is associated with a positive index of refraction in equ. 1 and with wave-like behavior required by conditions 2 and 4 above. Instabilities result from the fact that these wave regions are bounded by turning points. The situation is obscured somewhat by the fact that both corners and the steering level have been compressed into a single level in the highly idealized Helmholtz profile. It should be noted that the Helmholtz profile is physically unrealizable. However, if one insists on using discontinuous velocity profiles, one must also allow vorticity waves in the infinitesimal corners in order to be consistent. A proper approach to the Helmholtz profile, is to view it as the limit of a continuous profile: *i.e.*,

$$U=\lim_{\varepsilon\to 0} U \tanh(y/\varepsilon),$$

for example. It can be shown that stability results converge to the traditional Helmholtz results as $\varepsilon\to 0$. Moreover, this result is not at all sensitive to the precise form of the continuous profile provided that it converges to the Helmholtz profile. For the continuous profile, the existence of the wave regions, the turning points and the steering level are all clear. Indeed, this was the approach taken by Lindzen and Rosenthal (1983). The profile they considered was essentially that shown in Fig.7 of *T*, but with stratification and without β. They showed that waves existed in the corners (essentially vorticity waves – of which Rossby waves are an example) even as the corners approached discontinuities in U_y. The point is simply that the wavelength of the wave shrinks sufficiently fast as the corner shrinks – so that the corner region remains slowly varying with respect to the wave and wave concepts remain appropriate. The quantization provided by the turning points was shown, moreover, to correspond almost exactly to the calculated instabilities of the discontinuous profile. Similar analyses can be conducted for the discontinuities which are essential features of each of *T*'s examples. There is, however, little point in doing so; afterall, there are an infinite number of models one could generate in which corners provide wave regions. The analysis of any one of these cases is sufficient to explain how all of them work.

To be sure, the presence of wave regions beyond the corners (as is provided by β or N^2 in *T*) complicates matters. In *T*'s example 4 with stratification, gravity wave modes are present together with the *K-H* instability due to vorticity waves in the corners. The reflection coefficient that *T* calculated for region 2 (away from the corners) is for the gravity modes only. The over-reflection of the waves trapped at the corners was not considered by *T*, and it is the latter that gives rise to the instability found by *T* in his eigenvalue problem. The same criticism applies to *T*'s treatment of his example 1. In cases where the index of refraction is positive both in the corners and beyond the corners, the sharp change of index of refraction at the outer edge of the corners acts to strongly reflect (thus partially containing) the waves, and waves outside the corners can result from leakage from within the corners. This can lead to strange effects when studying the scattering properties of waves outside the corners. In the numerical study of waves in fluids with critical levels, one generally includes a small amount of damping (in the form of small, positive c_i). For over-reflection of the sort we have been discussing one always finds that over-reflection asymptotes towards its maximum value as $c_i\to 0$. However, Rosenthal and Lindzen (1983b) found for waves outside the corners, over-reflection sometimes maximized at a finite c_i. As Lindzen and Barker (1985) note, these cases seem to always be associated with instabilities involving waves partly contained at the corners. The over-reflection which maximizes at finite c_i is associated with the leakage of these unstable modes, kc_i being the growth rate.

Two of *T*'s results require special comment. In *T*'s second example (piecewise-linear shear layer on a β-plane), he finds no over-reflection for $-1<c<1$. This is entirely consistent with the results in Lindzen and Tung (1978); the entire region $y<y_c$ (where $U(y_c)=c$) is a single wave region wherein β-U_{yy} has not changed sign. Under these conditions, Lindzen and Tung (1978) showed that there could not be any over-reflection. The only place where $\beta-U_{yy}$ does change sign is in the corner at $y=d$ where $U=-1$, and thus the only way the conditions for over-reflection can be satisfied is for $c=-1$. *T* did not examine his equ. 3.4 for *R* at $c=-1$. In *T*'s other two cases (both involving Helmholtz profiles for *U*), it is observed that the Reynolds stress is continuous across the flow discontinuity. As Lindzen and Rosenthal (1983) note, this is no longer true at the critical levels in the continuous profiles to which the Helmholtz profile is an approximation. The discontinuity is associated with momentum exchanges within the shear

layer (including the corners). These exchanges appear to cancel as the two corners and the shear layer are all collapsed into a single level – leading in turn to the apparent continuity of Reynolds stress. This misleading feature is characteristic of such highly degenerate profiles as the Helmholtz profiles[3].

We turn finally to the reason for our contention that the existence of at least one wave region may be essential to all plane parallel flow shear instabilities. As discussed in Lindzen and Barker (1985) (and as we have already discussed in connection with condition 4), there are reasons to believe that both over-reflection and instability arise in the amplification of disturbances by the shear at the critical level via the Orr mechanism. This mechanism when applied to an initial perturbation tilted in a direction opposite to the shear, leads to growth for the finite time it takes the shear to rotate the disturbance tilt until it is in the direction of the shear; after this time, it causes the disturbance to decay. (The reader may find a particularly simple description of this mechanism in Boyd (1983). A more thorough discussion is given in Farrell (1987)) The only way to use this mechanism to provide continuous growth is to provide some means whereby a disturbance generated by the Orr mechanism can be turned around and returned to the critical level where it can provide a new "initial perturbation". As a practical matter, the only way a disturbance can be "turned around" is if it is in the form of a wave which can be reflected. This matter is discussed in detail in Lindzen (1987).

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[3] *T.* Matsuno has usefully noted that in shear instability the growth of disturbance energy comes solely from the conversion term, $u_z\overline{u'w'}$, which in the Helmholtz problem is undefined because the shear layer is infinitesmally collapsed.

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高橋正明：過剰反射のないシアー不安定について（気象集誌64巻793-80ページ）へのコメント

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表記論文において、高橋は、著者ら（Lindzen, Tung）が展開して来た「シアー不安定は過剰反射の結果と解釈できる」という考えに対して反例を示し、反論を試みている。このコメントにおいて、著者らは、高橋論文の個別結果に誤りはないが、解釈、とくに数学的簡便さのための理想化（不連続なシアーをもつ流れ）をしたときの解釈について高橋論文は不適切であることを指摘し、また高橋論文において強調されているシアー不安定のための条件は、原著者の主張を誤解していることを指摘する。結論として、高橋論文の見かけの反例にもかかわらず、前記の考えは依然として妥当である。