The Characteristic Rossby Frequency

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ABSTRACT

The characteristic Rossby frequency is defined for a fixed zonal wavenumber perturbation as the variational integral of the Rayleigh–Ritz method. It is a measure of the time scale of the disturbance. For a disturbance which locally has the shape of an eigenfunction but is not global in extent, the characteristic Rossby frequency is very close to the true eigenvalue, and additionally remains unchanged under linear inviscid dynamics. Results are presented for the shallow water equations, both with and without a mean zonal wind. The characteristic Rossby frequency of a wavenumber 1 perturbation having the shape of the second symmetric Rossby mode but confined to the Northern Hemisphere is close to the corresponding Rossby frequency. This finding is helpful in understanding the behavior of the observed wavenumber 1 pattern of January 1979, which propagated westward with nearly the pure Rossby frequency but was discernible only in the Northern Hemisphere (as discussed by Daley and Williamson).

1. Introduction

The availability of global analyses has sparked interest in the subject of global rotational normal modes (Rossby waves) and the extent to which they are present in the atmosphere. Recent papers include those by Ahlquist (1982), Hirota and Hirooka (1984), Lindzen et al. (1984), Salby (1984), Hirooka and Hirota (1985).

Of particular interest is the paper of Daley and Williamson (1985; hereafter referred to as DW), who examined an episode during January 1979 in which a large wavenumber 1 perturbation propagated westward with a period of roughly 19 days. They calculated the normal modes in the presence of the January 1979 mean zonal wind, taking into account both vertical and horizontal zonal wind variations, and found that the second symmetric rotational (external) mode had a theoretical period of about 17 days, close to the observed period of 19 days. They also found that the global observations had substantial projections onto both the *second symmetric* and *second antisymmetric* rotational eigenfunctions, and that *both* projections moved westward with nearly the same phase speed. While this westward propagating signal dominated the total observed wavenumber 1 signal in the Northern Hemisphere and was similar in shape to the second symmetric eigenfunction there, virtually no signal was discerned in the Southern Hemisphere. In particular, the signal was very weak at 60°S, where the eigenfunction of the mode is large.

An explanation for these findings was offered by Salby (1984), who pointed out that the structures of the second symmetric and second antisymmetric eigenfunctions are similar enough (in the Northern Hemisphere) so that a linear combination of the two functions could give results like the observed pattern: a response confined to the Northern Hemisphere. But this explanation cannot account for the fact that the projection onto the second *antisymmetric* eigenfunction in DW clearly moves with a frequency much closer to the eigenfrequency of the second *symmetric* mode (period 17–19 days) than to the eigenfrequency of the second antisymmetric mode (period of 28 days). This is just a reflection of the fact that the total disturbance moves with a well-defined period of about 19 days. Expanding in terms of (global) eigenfunctions is thus rather confusing in this case, for the atmosphere seems to be able to "recognize" the 19-day mode on the basis of its Northern Hemisphere structure alone.

In this paper we wish to show that while the observed confinement of the wavenumber 1 disturbance during January 1979 cannot be accounted for without consideration of the effects of forcing, damping and possibly a more complicated basic state, the close agreement between the frequency of the disturbance and that of the second symmetric global normal mode is to be expected on the basis of linear, inviscid dynamics. To do this we must take up the question of how much of the full global normal mode structure must be present in order for the atmosphere to recognize a mode and propagate it (zonally) at approximately the theoretical Rossby frequency.

We introduce the characteristic Rossby frequency, which is a measure of the frequency a perturbation moves with, and is relevant for initial conditions which have some normal mode structure but which may be localized. The advantage of the Rossby frequency lies in the fact that while the evolution of a wavenumber 1 perturbation can be rather complicated (as emphasized by Salby, 1984) and is sensitive to the precise initial meridional structure, the Rossby frequency gives the dominant zonal propagation (phase) speed and is less sensitive to the precise meridional structure. This will be illustrated by the use of the spectrum in section 3. The characteristic Rossby frequency is defined in section 2. It is calculated for a variety of perturbations having the shape of the second symmetric Rossby mode (locally) in the shallow water system in section 3. A brief discussion is given in section 4.

2. Characteristic Rossby Frequency

We consider perturbations of fixed zonal wavenumber s, and write

$$\mathbf{b} = \mathbf{B}e^{is\lambda}, \qquad (1)$$

where $\mathbf{b}$ is the appropriate state vector of the system. For example, in a nondivergent barotropic system, $\mathbf{b}$ would consist of values of the streamfunction at different latitudes, or equivalently, the values of the coefficients in an expansion of the streamfunction in associated Legendre polynomials. In the shallow water system, $\mathbf{b}$ would consist of values of the zonal wind u, the meridional wind v and the height h at different latitudes, or values of the coefficients in associated Legendre polynomial expansions. We may write the linearized equations of motion in the generalized form

$$\frac{\partial \mathbf{B}}{\partial t} = -i\mathsf{L}\cdot\mathbf{B}, \qquad (2)$$

where L is the appropriate linear operator. If a mean zonal wind is assumed, it enters in the definition of L. If we further assume that only one frequency (σ) is present,

$$\mathbf{B} = \hat{\mathbf{B}}e^{i\sigma t}, \qquad (3)$$

Eq. (2) becomes the familiar eigenvalue/eigenvector problem,

$$\mathsf{L}\cdot\hat{\mathbf{B}} = \sigma\hat{\mathbf{B}} \qquad (4)$$

whose nth eigenvalue and eigenvector are denoted as σ_n and $\hat{\mathbf{B}}_n$. In the absence of a mean flow, Eq. (4) is Laplace's tidal equation. This equation, and its modification when a mean zonal flow is present, have been extensively discussed in the literature (Longuet-Higgins, 1968; Ahlquist, 1982; Kasahara, 1976). There is no point in repeating the discussion here.

If we let $\langle\hat{\mathbf{B}}^{(1)}, \hat{\mathbf{B}}^{(2)}\rangle$ be the appropriate inner product of two arbitrary disturbances $\hat{\mathbf{B}}^{(1)}$ and $\hat{\mathbf{B}}^{(2)}$, then the ratio

$$F_R = \langle\hat{\mathbf{B}}, \mathsf{L}\cdot\hat{\mathbf{B}}\rangle / \langle\hat{\mathbf{B}}, \hat{\mathbf{B}}\rangle \qquad (5)$$

takes the value of σ_n if $\hat{\mathbf{B}} = \hat{\mathbf{B}}_n$. (In the case of the shallow water equations,

$$\langle\hat{\mathbf{B}}^{(1)}, \hat{\mathbf{B}}^{(2)}\rangle = \int_{-\pi/2}^{\pi/2} d\phi \cos(\phi)\{\hat{u}^{(1)*}\hat{u}^{(2)} + \hat{v}^{(1)*}\hat{v}^{(2)} + \hat{h}^{(1)*}\hat{h}^{(2)}\},$$

where $(\hat{u}^{(1)}, \hat{v}^{(1)}, \hat{h}^{(1)})$ and $(\hat{u}^{(2)}, \hat{v}^{(2)}, \hat{h}^{(2)})$ are the nondimensional zonal wind, meridional wind and height corresponding to $\hat{\mathbf{B}}^{(1)}$ and $\hat{\mathbf{B}}^{(2)}$.) We define F_R to be the characteristic Rossby frequency. The characteristic Rossby frequency has the property that it is a (relative) minimum for a true eigenfunction (see Friedman, 1966), so that a trial function which departs from a true eigenfunction by a small amount ϵ leads to errors only of order ϵ^2 in F. (This principle forms the basis of the Rayleigh–Ritz method of obtaining eigenvalues.)

We suggest that the integral (5) gives a valid measure of the characteristic frequency of a disturbance even when that disturbance does not consist of a single normal mode. This interpretation is strictly analogous to the formulation of the energy of a particle in quantum mechanics (see Landau and Lifshitz, section 20, page 58, 1965). It is also true that this measure of frequency is invariant in time under linear, inviscid dynamics. For, we can write an arbitrary wavenumber 1 disturbance $\mathbf{B}$ as

$$\mathbf{B} = \sum_n c_n\hat{\mathbf{B}}_n e^{i\sigma_n t}, \qquad (6)$$

where all the time dependence is in the exponential. Then

$$F_R = \sum_{n,m} c_n^* c_m e^{i(\sigma_m-\sigma_n)t} \times \langle\hat{\mathbf{B}}_n, \mathsf{L}\cdot\hat{\mathbf{B}}_m\rangle / \sum_{n,m} c_n^* c_m e^{i(\sigma_m-\sigma_n)t}\langle\hat{\mathbf{B}}_n, \hat{\mathbf{B}}_m\rangle$$

which becomes, using the fact that $\hat{\mathbf{B}}_n$ is an eigenvector of L and the orthogonality of $\hat{\mathbf{B}}_m$, $\hat{\mathbf{B}}_n$,

$$F_R = \sum_n |c_n|^2\sigma_n / \sum_n |c_n|^2,$$

a constant in time. The variational principle guarantees that the value of the integral (characteristic Rossby frequency) is robust to changes in the perturbation $\mathbf{B}$ when $\mathbf{B}$ is nearly an eigenfunction, in the sense discussed earlier. When the perturbation is localized, so are the integrals in (5), indicating that the behavior of the basic state plays no role in regions where the wave amplitude is zero. In section 3, we explore what happens when $\mathbf{B}$ is far from an eigenfunction, but nevertheless retains some information about the meridional scale of the eigenfunction.

3. Numerical results

In this section we evaluate the characteristic Rossby frequency for a family of perturbations which are related to the second symmetric Rossby (external) mode for zonal wavenumber 1. We work with the linearized shallow water equations (with the equivalent depth set to 10 km) and give results both with and without mean zonal winds.

We first consider the case with no mean zonal wind. The eigenfunctions of the normal mode problem are then known as Hough functions (see, for example, Kasahara, 1976), and consist of vector functions of (u, v, h), where u is the zonal velocity, v the meridional velocity, and h the height field. The family of perturbations considered was generated by multiplying the second symmetric rotational eigenfunction by $f(\phi)$, defined by

$$f(\phi) = \exp\{-[(\phi - \phi_0)/a]^4\}, \qquad (7)$$

where ϕ is latitude (in degrees), ϕ_0 the central latitude of the pulse (in degrees) and the width parameter a is also given in degrees. Since $\exp(-1.2)^4$ is ~ 0.1, the perturbation is effectively contained between latitudes $(\phi_0 - 1.2a)$ and $(\phi_0 + 1.2a)$.

The characteristic Rossby frequency was calculated for members of this family and the results plotted in the form of a normalized frequency shift in Fig. 1. The frequency shift is defined as the difference between the characteristic Rossby frequency and the frequency of the second symmetric rotational normal mode (with no mean wind). It is normalized by the difference between the frequency of this normal mode and its closest neighbor (in frequency), the second antisymmetric rotational mode. *Thus, when the normalized frequency shift is much less than 1, the pulse has a frequency closer to that of the second symmetric mode than to any other normal mode frequency.* In Fig. 1 we see that for broad pulses (width parameter greater than 35°), the characteristic Rossby frequency is very close to the normal mode frequency for all locations of the pulse (all values of ϕ_0). Thus when the perturbation extends $35 \times 1.2 = 42°$ or more to either side of the central latitude, the meridional scale of the global eigenfunction is apparently well captured. A pulse centered at high latitudes (60°N or 60°S) having a width parameter a of 60° penetrates very little into the opposite hemisphere. This is relevant for the observations of DW, in which the observed disturbance was confined to the Northern Hemisphere but propagated with a (global) normal mode frequency. A closer connection with the observations will be made in the context of the calculations which take into account mean zonal winds.

There is a stronger dependence of the normalized frequency shift on the latitude of the pulse center when the pulse is narrow (width parameter less than 30°). In order for a narrow pulse to capture the essential meridional structure of the mode, it must be centered near the zero crossings of the eigenfunction. For example, a small frequency shift is seen for pulses centered in the neighborhood of 10° to 20°. A narrow pulse centered at 15°N would extend to roughly 30° or 35°N, close to where the true eigenfunction (Hough function) has its one zero crossing for h and v in the Northern Hemisphere. Thus, when the meridional scale of the disturbance is specified correctly, the characteristic frequency becomes quite close to the eigenvalue, even when the function as a whole does not resemble the eigenfunction.

In order to make a closer comparison to the observations of DW, we have included the effects of the mean zonal wind. This was done by using the 500 mb zonally averaged wind (averaged over December–February of 1979) in the shallow water system. This level

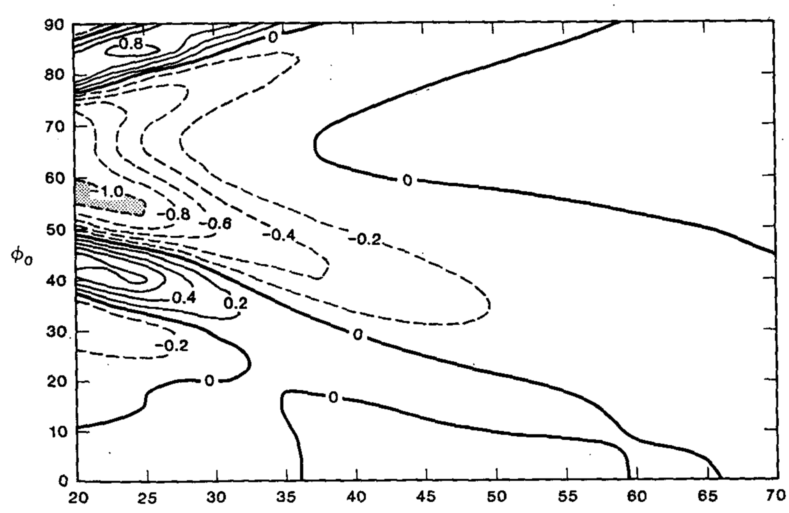

FIG. 1. The normalized frequency shift of a family of perturbations related to the second symmetric rotational mode in the case of no zonal wind. The abscissa is the envelope half-width a (see the text) and the ordinate is the center of the envelope (ϕ_0), both given in degrees. The contour interval is 0.2, and negative contours are dashed. Shading indicates regions where the frequency shift exceeds 1.0 in absolute value.

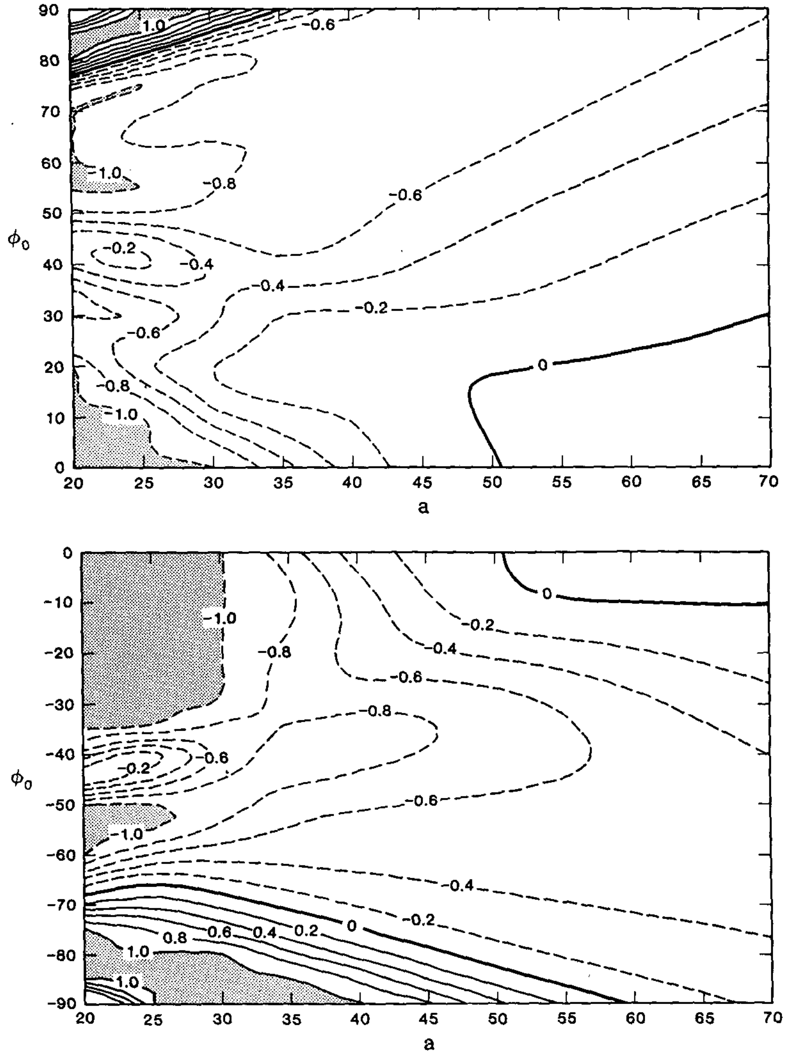

FIG. 2. As in Fig. 1, but for a family of perturbations related to the second symmetric mode with a mean zonal wind taken from the winter 1979 zonal wind average at 500 mb. Panel (a) gives results for the center (ϕ_0) in the Northern Hemisphere and Panel (b) for (ϕ_0) in the Southern Hemisphere.

was chosen because the periods of the second symmetric mode (18.7 days) and the second antisymmetric mode (29.5 days) in this calculation are close to the periods given from both the more complete normal mode calculation and the observations reported by DW. Figure 2 shows the difference between the Rossby characteristic frequency and that of the second symmetric eigenfrequency, normalized as before. To aid in the comparison with DW, we note that a normalized frequency shift of −0.4 corresponds to a period of about 16 days, and one of +0.4 corresponds to a period of about 22 days. It is seen from Fig. 2 that the normalized frequency shift is quite small for a wide range of choices of the perturbation center (ϕ_0) and the perturbation width a. Of immediate relevance to DW's results is the fact that a disturbance centered at 60°N with a width of about 65° (which is thus confined to the Northern Hemisphere) has a characteristic Rossby frequency corresponding to about 16 days (normalized shift of −0.4). This is fairly close in frequency to that reported by DW, which corresponds to a period of 19 days. Note that this result is almost symmetric about the equator; a perturbation of width 65° centered at 60°S also has a small normalized frequency shift, although not as small as when the center is at 60°N.[1]

The characteristic Rossby frequency can be thought of as an "average" frequency for a particular distur-

[1] Since the zonal wind is not symmetric about the equator, neither is the characteristic Rossby frequency if considered as a function of ϕ_0. The degree of asymmetry is small, however.

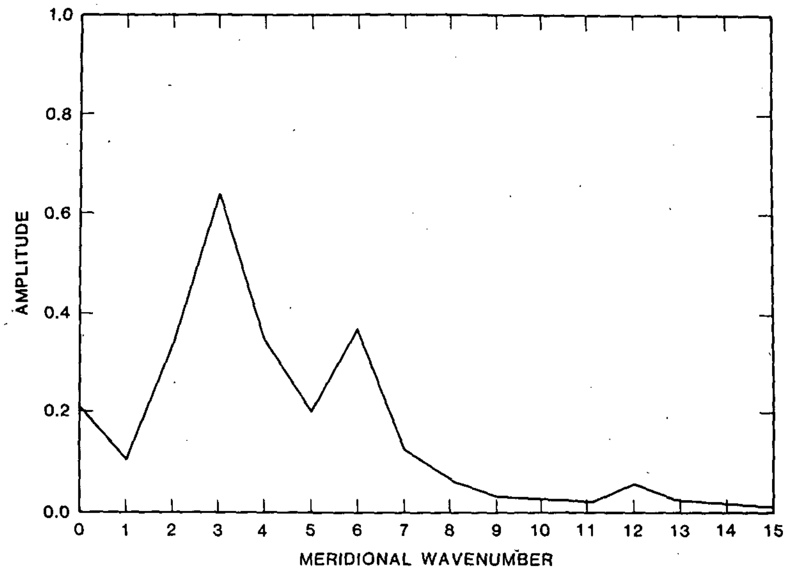

FIG. 3. The normalized amplitude of the projection of a "truncated" mode (of width 30° centered at 30°N) on the full eigenmodes in the case of no zonal wind. The correspondence between meridional wavenumber and other mode characteristics is given in Table 1. Meridional wavenumber 3 corresponds to the second symmetric (18.7 day) Rossby mode.

bance. While it is true that this average is close to the Rossby mode frequency for a disturbance which is suitably wide meridionally, will such a disturbance actually propagate at a frequency near this "average" frequency, or are so many frequencies involved that the average is just a statistical residue—the center of a wide distribution? In this regard it should be noted that although a "truncated" normal mode may travel with the eigenfrequency, the decomposition of the "truncated" normal mode in terms of untruncated modes leads to a spectrum in terms of the untruncated modes.[2] An example of such a decomposition is portrayed in Fig. 3 for the second symmetric Rossby mode (in the absence of zonal winds), truncated by multiplication by the function f of Eq. (7), with $\phi_0 = 30°$ and width $a = 30°$. What is plotted is the normalized amplitude c_n of each normal mode, given by $c_n/\sum_m |c_m|^2$. We see that the amplitude peaks at the meridional wavenumber 3 of the mode which was truncated, and that in the neighborhood of this peak the spectrum is nearly symmetric. Now the two adjacent meridional wavenumbers (2 and 4) have frequencies different from meridional wavenumber 3—but the pair will appear to oscillate with their average frequency, which is very nearly the frequency associated with meridional wavenumber 3. Note that the amplitude of the projection is about 0.6 for the second symmetric mode rather than 1.0. Thus, had we analyzed this "truncated" mode by projecting it onto the global normal modes (as in Lindzen et al., 1984), we would have obtained an amplitude that was only about 60% of the local amplitude of the mode. This may explain why analyses at single latitudes based on time series (as in Madden, 1979; Madden and Labitzke, 1981) show larger amplitudes than normal mode projections (as in Lindzen et al., 1984).

Another example, more directly relevant to the observations of DW, is given in Table 1. Here we show the spectrum of a disturbance which is obtained by multiplying the second symmetric Rossby eigenfunction (in the presence of 500 mb winter zonal winds from 1979) by $f(\phi)$, with center ϕ_0 at 60°N and widths of both 30° and 60°. The spectrum S is given only for four modes, as it is extremely weak for all others. The table clearly indicates that the spectrum is peaked at the 18.7 day frequency of the second symmetric rotational mode, and that the peak is sharper for the wider disturbance. The spectrum for the narrow ($a = 30°$) disturbance is less symmetric about the peak than in Fig. 3, and is skewed towards more negative frequencies (faster westward propagation). This corresponds to the normalized frequency shift of about −0.8 seen in Fig. 2 for $\phi_0 = 60°$ and $a = 30°$. The spectrum of the wider disturbance is less skewed, leading to a normalized frequency shift closer to zero in Figure 2. These results clearly show that for wide disturbances the spectrum is dominated by the mode whose shape the disturbance locally has, so that the characteristic Rossby frequency is truly characteristic.

[2] In the language of Eq. (6), the spectrum S_n is $S_n = |c_n|^2/\sum_m |c_m|^2$.

TABLE 1. The spectrum of a perturbation consisting of the second symmetric rotational normal mode (in the presence of 500 mb winter-averaged 1979 winds) multiplied by an envelope centered at 60°N, with widths of 30° or 60°. Only the four modes with substantial power are shown. The meridional index of each mode is given in parentheses. The frequency is in units of twice the earth's angular velocity, the period is given in days, and the power is normalized so that it equals 1.0 when summed over all modes.

Mode (meridional wavenumber)	Freq.	Period (days)	Power 30°	Power 60°
First symmetric (1)	−0.1011	4.9	0.118	0.028
First antisymmetric (2)	−0.0517	9.7	0.231	0.200
Second symmetric (3)	−0.0267	18.7	0.425	0.620
Second antisymmetric (4)	−0.0169	29.5	0.159	0.109

4. Discussion

The purpose of the foregoing (simple) calculations was to show how much modal structure in the meridional direction need be present in order for the atmosphere to, in some sense, "recognize" a normal mode. The calculations suggest that the second symmetric rotational external mode (which has a period near 19 days in the atmosphere) can be fairly well defined by a disturbance which has no amplitude in southern midlatitudes.

This has a direct application to the results of DW, and shows that the close agreement between the frequency of the disturbance and that expected of the second symmetric global normal mode is to be expected on the basis of linear, inviscid dynamics.

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