The Eady Problem for a Basic State with Zero PV Gradient but $\beta \neq 0$

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ABSTRACT

The classic Eady problem is modified to include $\beta \neq 0$, but with the basic distributions of temperature and zonal flow adjusted to preserve zero meridional gradients of basic-state potential vorticity in the fluid interior. Much of the mathematical simplicity of the classic problem is retained; however, the results differ in important ways. Specifically, the instability now has a long-wave cutoff in addition to the traditional short-wave cutoff. The former is associated with the fact that the phase speeds of the edge waves begin to differ so much as wavenumber is reduced that the two edge waves can no longer interact in order to form unstable modes. For the unstable modes, this manifests itself in that the steering level for unstable modes is always below the middle of the fluid and approaches the lower boundary near the long-wave cutoff. Relatedly, the amplitude of the unstable geopotential perturbations is larger at the upper boundary than at the lower boundary. Finally, below the long-wave cutoff, one of the neutral waves has a phase speed that becomes increasingly easterly as wavenumber decreases. This allows a resonant response to planetary-scale stationary forcing.

1. Introduction

The Eady problem (Eady 1949) is one of the classic treatments of baroclinic instability. By ingenious simplification, Eady managed to produce a problem whose mathematics is almost trivial. In contrast to the more general Charney problem (Charney 1947) the solution involves simple exponentials, rather than confluent hypergeometric functions, and is free of singularities. The simplifications consist basically in using a quasigeostrophic system (just as Charney did) on an f plane (in contrast to Charney's use of a β plane) together with a Boussinesq fluid with lower and upper rigid boundaries (unlike Charney). The basic state (for the troposphere at least) consisted in a zonal flow with constant shear and Brunt–Väisälä frequency (again in common with Charney). However, in contrast to the Charney problem, for a Boussinesq fluid on an f plane such a basic state is associated with zero meridional gradient of potential vorticity. This is the feature at the heart of the simplicity of the Eady problem. Charney and Stern (1962) showed that a necessary condition for baroclinic instability is that some surface exist across which the meridional gradient of potential vorticity (PV) changes sign. The sign change in the Eady problem arises from δ-function contributions to PV gradients at the rigid surfaces (Bretherton 1966; Lindzen and Tung 1978; Lindzen 1990).

Although the simplifications in the Eady problem may seem severe, the observed atmosphere is indeed characterized by a troposphere with small PV gradients (Pfeffer 1981; Hoskins et al. 1985), which are observationally indistinguishable from zero (Sun and Lindzen 1994), and concentrated PV gradients in the neighborhood of the tropopause. To be sure, it is an important matter to understand how the atmosphere reaches this state, but given this state, the Eady problem appears more relevant than one might have initially supposed. We will review the results of the Eady problem in section 3. One significant difference between the Eady problem and the Charney problem is the existence of a short-wave cutoff for instability in the former. The horizontal wavelength corresponding to the cutoff is scaled by the depth of the fluid. Lindzen (1993) noted that the existence of this short-wave cutoff suggests a way in which baroclinic instability could act to neutralize the atmosphere with respect to baroclinic instability. The jetlike character of the basic flow establishes a minimum horizontal scale for baroclinic instabilities (Ioannou and Lindzen 1986, 1990). If the eddies wipe out PV gradients up to some *sufficient* height, then this minimum scale will correspond to the short-wave cutoff, and the fluid will be neutralized. Lindzen (1993) estimated that this height would roughly coincide with the observed tropopause height.

There are, of course, many serious questions about the specific relevance of the Eady problem to the somewhat more complicated situation occurring in nature. In the present paper, we focus on one specific difference. Namely, how does a problem where $\beta \neq 0$ and where zero PV gradients are achieved by modifying the

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distributions of U and T differ from the classical Eady problem?

In section 2 we will briefly review the basic equations for the Eady problem as modified for the $\beta \neq 0$ case. We also go over the modification of the basic state required to produce zero PV gradients. We then go over the formal solution to the modified problem. In section 3, we look at the results for the modified instability problem, comparing results with classical Eady results and with observed eddy behavior. One important result we find is that the modified problem not only reproduces the short-wave cutoff of the classical problem (which is not surprising) but also possesses a long-wave cutoff. Beyond this cutoff there is a pair of neutral waves. The phase speed of one of these waves approaches $-\infty$ as the horizontal wavenumber approaches zero. This suggests that, in the presence of stationary forcing, a resonant response is possible. This is discussed in section 4. Section 5 includes concluding remarks.

2. Equations and solutions

The basic equation for both the Charney and Eady problems is the linearized quasigeostrophic equation for conservation of pseudo-potential vorticity where density variation with height is approximated by an exponential with constant scale height, H (for details see Lindzen 1990; Pedlosky 1987):

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)q' + v'\frac{\partial \bar{q}}{\partial y} = 0, \qquad (1)$$

where U is the basic zonal flow and $\bar{q}$ is the basic distribution of pseudo-potential vorticity, v' the perturbation meridional velocity, y the meridional direction, x the zonal direction, and t time. Here v' and q' are related to the perturbation geopotential Φ' by

$$v' = \frac{1}{f_0}\frac{\partial \Phi'}{\partial x} \qquad (2)$$

$$q' = \frac{1}{f_0}\left(\frac{\partial^2\Phi'}{\partial x^2} + \frac{\partial^2\Phi'}{\partial y^2}\right) + e^{z^*}\frac{\partial}{\partial z^*}\left(\frac{f_0}{H^2N^2}e^{-z^*}\frac{\partial\Phi'}{\partial z^*}\right), \qquad (3)$$

where N is the basic state's Brunt–Väisälä frequency; f_0 is the Coriolis parameter on a β-plane channel; and z, height, $\equiv Hz^*$. The meridional gradient of the basic pseudo-potential vorticity is related to U and N^2 by

$$\frac{\partial \bar{q}}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \beta - e^{z/H}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}e^{-z/H}\frac{\partial U}{\partial z}\right). \qquad (4)$$

At rigid horizontal surfaces, the condition that the vertical velocity equal zero leads to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial\Phi'}{\partial z} - \frac{\partial\Phi'}{\partial x}\frac{\partial U}{\partial z} = 0. \qquad (5)$$

In both Charney and Eady problems, one ignores the barotropic curvature of U in Eq. (4). In the Eady problem there is a rigid upper lid at $z = h$, and the Boussinesq approximation is equivalent to taking $h \ll H$. Moreover, $\beta = 0$. Then Eqs. (3) and (4) become

$$q' = \frac{1}{f_0}\left(\frac{\partial^2\Phi'}{\partial x^2} + \frac{\partial^2\Phi'}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\frac{\partial\Phi'}{\partial z}\right) \qquad (6)$$

and

$$\frac{\partial \bar{q}}{\partial y} = -\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}\right). \qquad (7)$$

Note that when N^2 and U_z are constant, Eq. (7) automatically reduces to $\bar{q}_y = 0$. Finally, we restrict ourselves to normal-mode instabilities where

$$\Phi' = \Phi'(z)\sin(ly)e^{ik(x-ct)}.$$

For the classical Eady problem, Eqs. (1) and (3) reduce to

$$(U - c)\left(\frac{\partial^2\Phi'}{\partial z^2} - \frac{k^2 + l^2}{f_0^2/N^2}\Phi'\right) = 0, \qquad (8)$$

whose solutions are simple hyperbolic functions. In the following subsection we will consider the more general case where $\beta \neq 0$, and N^2 and U_z are adjusted to produce $\bar{q}_y = 0$.

a. Modified Eady problem

Now that we have β, $\bar{q}_y = 0$ requires that the basic state satisfy

$$\beta - \frac{d}{dz}\left(\frac{f_0^2}{N^2}\frac{dU}{dz}\right) = 0. \qquad (9)$$

We may choose to satisfy Eq. (9) by altering N^2, U, or both. If we alter only U, then Eq. (8) remains correct; only the boundary conditions at $z = 0$ and h are affected. However, if we alter N^2, Eq. (8) is changed as well. Let us first consider the case where we keep $dU/dz = m_0$ = const. Solving Eq. (9) for N^2, we get

$$N^2 = N_0^2 \Big/ \left(1 + \frac{\beta z}{\epsilon_0 m_0}\right), \qquad (10)$$

where $\epsilon_0 \equiv f_0^2/N_0^2$, and N_0 is the value of N at $z = 0$. If we wish to achieve only part of the adjustment to $\bar{q}_y = 0$ by changing N^2, we can replace Eq. (10) with

$$N^2 = N_0^2 \Big/ \left(1 + \alpha\frac{\beta z}{\epsilon_0 m_0}\right), \qquad (10a)$$

where α is the fraction of adjustment due to changing N^2. From Eq. (9), we must then modify U so that

$$\frac{dU}{dz} = m_0 \frac{1 + \dfrac{\beta z}{\epsilon_0 m_0}}{1 + \alpha \dfrac{\beta z}{\epsilon_0 m_0}}. \qquad (10b)$$

With either Eq. (10) or (10a) for N^2, the solutions to Eqs. (1) and (3) can be expressed in terms of the zeroth-order Bessel functions I_0 and K_0. The solutions, which are straightforward to obtain, will be given here primarily in order to present the nondimensionalization that will be used in discussing the results. It should be noted that our results are almost the same regardless of how we adjust to $\bar{q}_y = 0$. We will, therefore, rather arbitrarily select $\alpha = 0.5$. We have also taken $h = 10$ km. It turns out that the adjustments to T and U called for are rather small. This is illustrated for characteristic values in Fig. 1.

b. Solutions to classic and modified Eady problems

For the classic Eady problem, we nondimensionalize lengths by h and velocities by $U(h) = mh$; $U(0)$ is taken to be zero; and $m = dU/dz =$ const. Nondimensional quantities are generally represented by ($\tilde{}$). Solutions are of the form

$$\Phi' = A \cosh(\mu\tilde{z}) + B \cosh(\mu\tilde{z}), \qquad (11)$$

where A and B are constant coefficients, and

$$\mu^2 \equiv \frac{h^2(l^2 + k^2)}{\epsilon}.$$

The boundary conditions are simply

$$\tilde{c}\frac{d\Phi'}{d\tilde{z}} + \Phi' = 0 \quad \text{at} \quad \tilde{z} = 0 \qquad (12a)$$

$$(\tilde{c} - 1)\frac{d\Phi'}{d\tilde{z}} + \Phi' = 0 \quad \text{at} \quad \tilde{z} = 1. \qquad (12b)$$

One substitutes (11) into (12a) and (12b), and obtains two homogeneous linear equations for A and B. The solvability condition is simply that the determinant of the coefficients of A and B equal zero. The resulting equation is a quadratic in $\tilde{c}$, and the two solutions are functions of μ. Given $\tilde{c}$, one also obtains Φ' to within an arbitrary factor.

The procedure is almost identical in the more general case where N^2 is given by (10a), but the formal solutions are clumsier. As before, lengths are scaled by h and speeds by $U(h)$. The solution to $q' = 0$ is now

$$\Phi' = AI_0(u) + BK_0(u), \qquad (13)$$

where I_0 and K_0 are modified Bessel functions of order zero (Abramowitz and Stegun 1964), and

$$u = 2\frac{\mu}{\alpha\tilde{B}}\sqrt{1 + \alpha\tilde{B}\tilde{z}},$$

where

$$\tilde{B} \equiv \frac{\beta h}{\epsilon_0 m_0}, \quad \mu^2 \equiv \frac{h^2(k^2 + l^2)}{\epsilon_0},$$

$$\epsilon_0 = \frac{f_0^2}{N_0^2}, \quad \text{and} \quad m_0 = \left.\frac{dU}{dz}\right|_{z=0}.$$

The boundary conditions are now

$$\tilde{c}\Phi'_{\tilde{z}} + \tilde{m}_0\Phi' = 0 \quad \text{at} \quad \tilde{z} = 0 \qquad (14a)$$

$$(\tilde{c} - 1)\Phi'_{\tilde{z}} + \tilde{m}_1\Phi' = 0 \quad \text{at} \quad \tilde{z} = 1. \qquad (14b)$$

Here

$$\tilde{m}_0 = \frac{hm_0}{U(h)}, \quad \text{and} \quad \tilde{m}_1 = \frac{hm_1}{U(h)},$$

$$\text{where} \quad m_1 = \left.\frac{dU}{dz}\right|_{z=h},$$

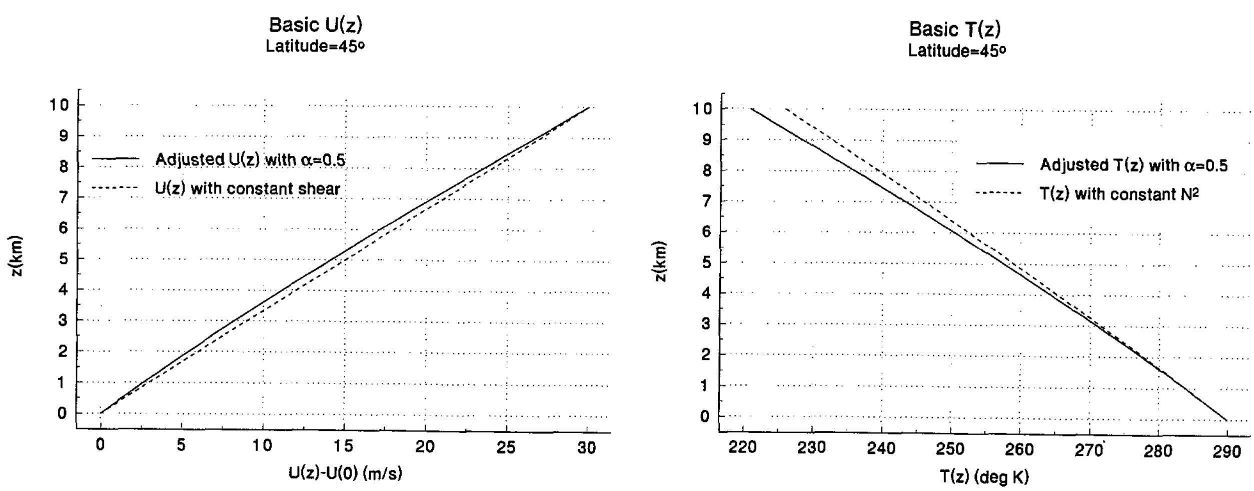

FIG. 1. Distributions of U and T corresponding to constant shear and N^2 (as used in the classic Eady problem), and basic states modified to produce $q_y = 0$ in the presence of β.

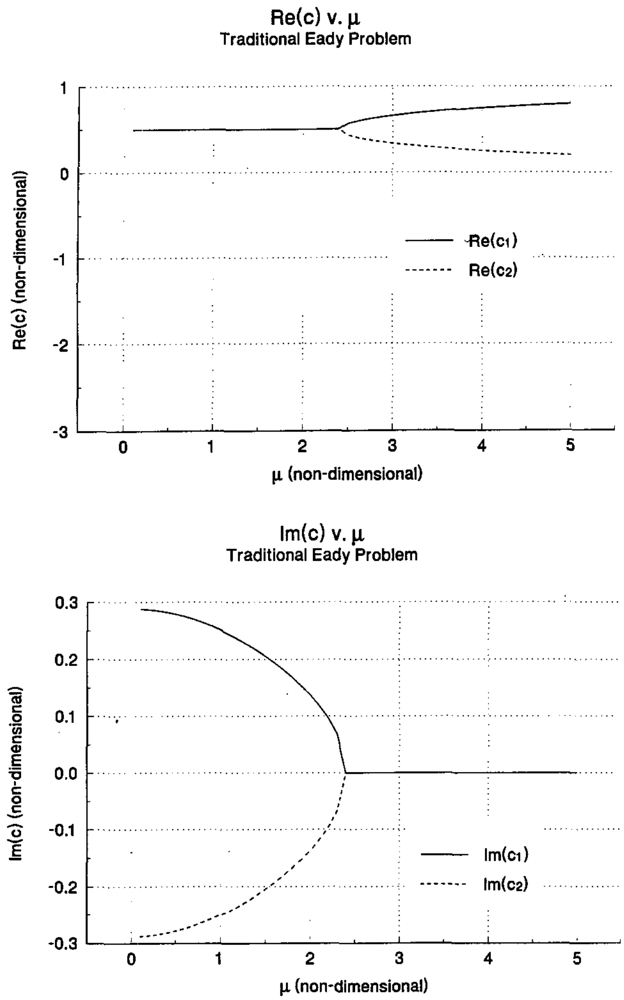

FIG. 2. Real and imaginary components of $\tilde{c}$ as functions of total (nondimensional) wavenumber μ for the classic Eady problem.

and

$$\Phi'_{\tilde z} = (AI_1(u) - BK_1(u))\frac{\mu}{\sqrt{1+\alpha\tilde{B}\tilde{z}}}. \qquad (15)$$

Despite the formal increase in complexity, the boundary conditions still lead to a quadratic equation in $\tilde{c}$, which yields two solutions that are again functions of μ (now slightly differently defined), and for these solutions we can again determine Φ' to within an arbitrary factor.

3. Results

The results for $\tilde{c}(\mu)$, appropriate to the classic Eady problem, are shown in Fig. 2. There is a short-wave cutoff for instability given by

$$\frac{\mu_c}{2} = \coth\left(\frac{\mu_c}{2}\right),$$

whose solution is

$$\mu_c \approx 2.3994.$$

These results are thoroughly discussed in standard texts. They are repeated here for purposes of convenient comparison with results from the modified Eady problem. The points we wish to stress are the existence of the short-wave cutoff, the absence of a long-wave cutoff, that the steering level for unstable waves is $\tilde{z} = 0.5$, and that the eigenfunctions (not illustrated) are symmetric about $\tilde{z} = 0.5$ and the functions vary little with wavenumber.

Figure 3 shows the real and imaginary components of c as a function of μ for the modified Eady problem. There is now both a short-wave and a long-wave cutoff. Moreover, the steering level is always below $\tilde{z} = 0.5$.

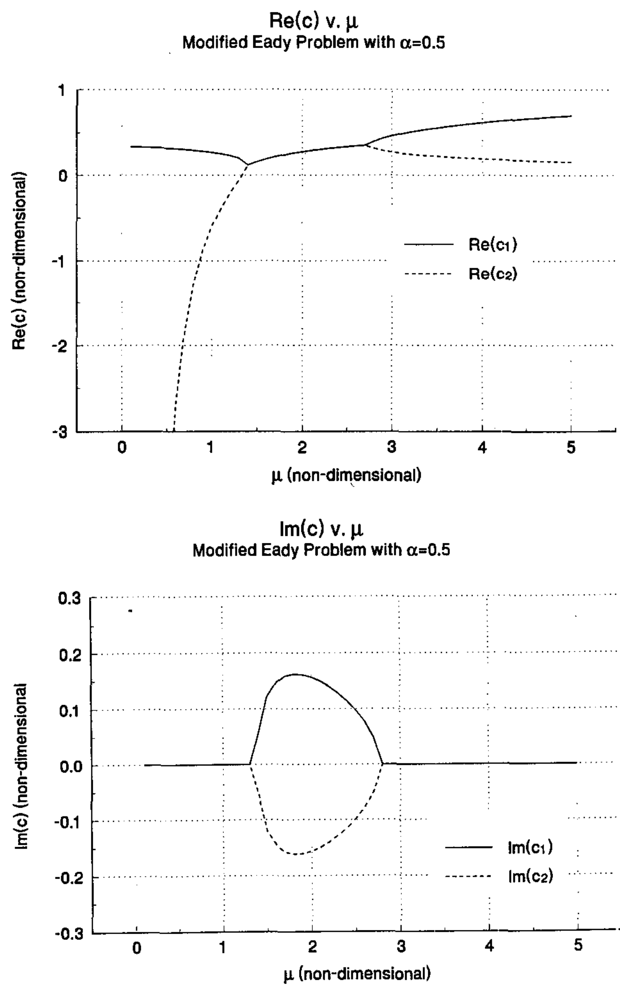

FIG. 3. Real and imaginary components of $\tilde{c}$ as functions of total (nondimensional) wavenumber μ for the modified Eady problem.

Indeed, as one approaches the long-wave cutoff the steering level for instability approaches $\tilde{z} = 0$. As one moves below the long-wave cutoff, the solution again splits into two neutral modes. In contrast to the neutral modes above the short-wave cutoff, one of the neutral modes actually has strong easterly phase speeds that approach $-\infty$ as μ approaches zero. With respect to instabilities, one must not overemphasize the long-wave cutoff since μ is a total wavenumber, and the presence of jet structures in actual zonal flows tends to introduce a meridional wavenumber of about $l = 2$. Thus, in terms of zonal wavenumber, there is no long-wave cutoff for instabilities. However, such considerations are no longer relevant to neutral modes. Here, the geometry of the earth is more relevant to establishing the meridional wavenumber. For a meridional wavenumber corresponding to a half wavelength of the radius of the earth, the nondimensional meridional wavenumber is ~0.5. Thus, stationary forcing can, in principle, lead to resonant responses. We will return to this matter in the next section.

Returning to the unstable modes, Fig. 4 shows the amplitude of the eigenfunctions for the most unstable mode, for a mode near the short-wave cutoff, and for a wave with zonal wavenumber near zero (assuming $l = 2$). We see that the minimum in amplitude is now below $\tilde{z} = 0.5$, and that the amplitude is greater at $\tilde{z} = 1$ than at $\tilde{z} = 0$. If one looked at the amplitudes for μ near its long-wave cutoff, the amplitude at $\tilde{z} = 1$ would be much larger than at $\tilde{z} = 0$ but, as we have noted, it is unrealistic to expect unstable modes with sufficiently large meridional scales to permit one to reach such small values of μ.

4. Stationary wave resonance

As we noted in the previous section, the modified Eady problem, in distinct contrast to the classic Eady problem, allows for resonant response to stationary forcing. In order to examine this point, we replace the lower boundary condition (5) with

$$\frac{\partial \Phi'}{\partial z} - \Phi' \frac{dU}{dz} = -h_0 U \quad \text{at} \quad z = 0, \qquad (16)$$

where $c = 0$ by definition for stationary waves and

$$h = h_0 e^{ikx} \cos(ly)$$

represents stationary forcing by surface orography. For the following results we will take $l = \pi/a$, where a is the radius of the earth, and $h_0 = 100$ m regardless of wavenumber. For convenience we use $|\Phi'(0.5)|$ as a measure of the response. Our procedure will be to allow $U(0)$ to vary and calculate the response as a function of k for each choice of $U(0)$. In Fig. 5 we show the results for the classic Eady problem, and in Fig. 6 for the modified Eady problem. We see no evidence of resonance, and the response decreases for all wavenumbers as $U(0)$ approaches zero. However, for the modified Eady problem, there is clearly resonance between $k = 3$ and 4. The precise position of the resonant response depends, as would be expected, on $U(0)$. While the possibility of stationary wave resonance for realistic choices of $U(0)$ is of great interest, it is altogether possible that the modified Eady problem with its rigid upper lid may prove misleading in this regard. We will separately publish a reassessment of this issue using an unbounded modification of the traditional Charney–Drazin problem (Charney and Drazin 1961). The modification consists in adjusting the basic state to eliminate or reduce potential vorticity gradients in the interior while concentrating them in the neighborhood of the tropopause. The results display resonant behavior in the more realistic problem as well.

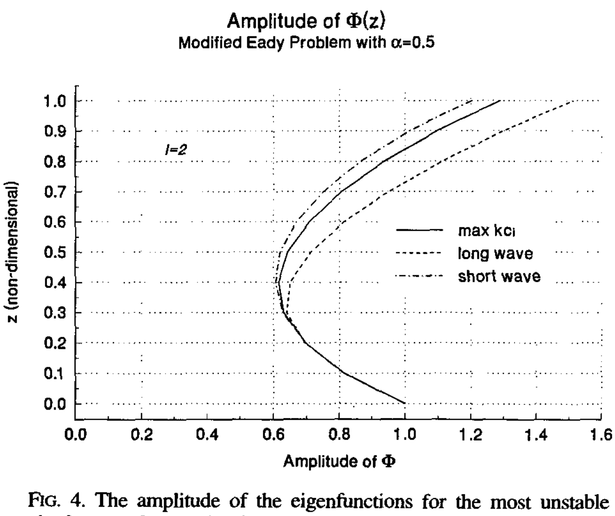

FIG. 4. The amplitude of the eigenfunctions for the most unstable mode, for a mode near the short-wave cutoff, and for a wave with zonal wavenumber near zero (assuming $l = 2$) for the modified Eady problem.

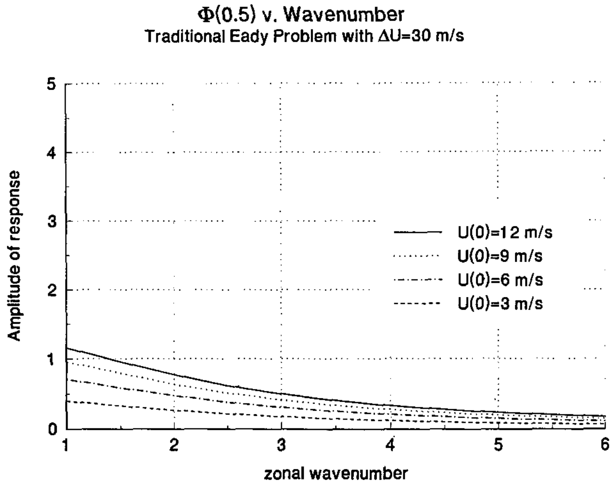

FIG. 5. A characteristic measure of response, $|\Phi'(0.5)|$, to topographic forcing as a function of zonal wavenumber k for various choice of surface basic flow $U(0)$. For the classic Eady problem.

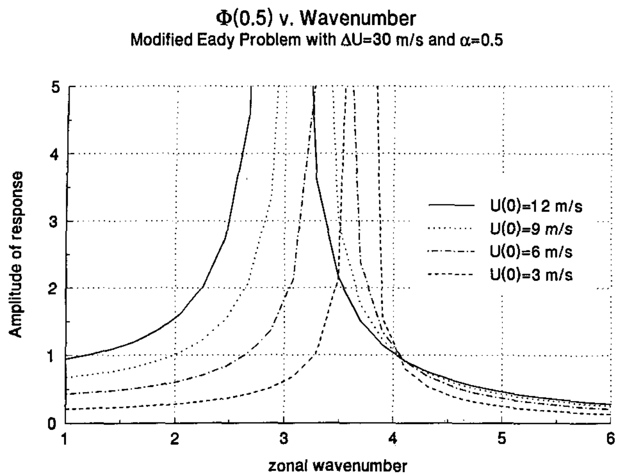

FIG. 6. A characteristic measure of response, $|\Phi'(0.5)|$, to topographic forcing as a function of zonal wavenumber k for various choice of surface basic flow $U(0)$. For the modified Eady problem.

5. Concluding remarks

The results in this paper confirm that the short-wave cutoff of the classic Eady problem remains when one allows $\beta \neq 0$ and modifies the basic state to achieve $q_y = 0$. However, the modification of the original problem leads to unstable eigenmodes with significant differences from the classic solutions. In particular, the steering level for unstable modes moves toward the lower boundary consistent with both observations and more general instability studies (Charney 1947; Green 1960). The unstable eigensolutions are no longer symmetric about the middle level, and at least for the perturbation geopotential, the eigensolutions are somewhat stronger at the top than at the bottom. Instabilities consisting in upper- and lower-level disturbances are reminiscent of Petterssen's Type B development (Petterssen and Smebye 1971). Because of the inclusion of β, upper and lower boundary waves tend, as $\mu \to 0$, to have dramatically different phase speeds. The new long-wave cutoff appears to result from the fact that at sufficiently small values of μ the boundary waves can no longer travel together sufficiently to interact. This is in contrast to the short-wave cutoff, which is associated with the boundary waves becoming so closely confined to the boundaries that they are no longer able to interact. The existence of neutral long waves with zonal phase speeds between small westerly values and infinite easterly values relative to the basic zonal flow at the surface leads to resonant response to stationary forcing at any reasonable value of the surface wind. Such a possibility could have importance for phenomena such as blocking.

Note added in proof. It has been pointed out to the author that some similar calculations were made by Grotjahn (1979).

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