Equatorial Planetary Waves in Shear: Part I

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ABSTRACT

The method of multiple scales is applied to the problem of the propagation of internal equatorial Yanai and Kelvin waves through shear. Analytic formulas for the various wave fields are obtained, and the role of dissipation and the behavior of momentum and energy fluxes are investigated. Results are compared with both numerical calculations and with observations. From the latter, we are able to estimate the dissipative time scale in the tropical stratosphere.

On the basis of the above findings, the Lindzen-Holton model of the quasi-biennial oscillation is modified.

1. Introduction

For several years now, long-period (5–12 days), planetary-scale internal waves confined to the neighborhood of the equator have been observed (Yanai and Maruyama, 1966; Maruyama, 1967; Wallace and Kousky, 1968a, b; Kousky and Wallace, 1971). Such waves were predicted by Lindzen (1967) and further described for simple shear-free atmospheres by Lindzen and Matsuno (1968) and Holton and Lindzen (1968).

In Lindzen and Holton's (1968) theory of the quasi-biennial oscillation such waves play an essential role as transporters of mean zonal momentum. Lindzen and Holton assumed the existence of internal equatorial waves with a wide range of zonal phase speeds whose momentum was transferred to the mean flow through critical level absorption.[2] Critical level absorption for internal gravity waves had been analyzed in some detail by Booker and Bretherton (1967) for a non-rotating atmosphere and by Jones (1967) for a plane rotating atmosphere. [Dickinson (1970) has since analyzed the critical surface absorption of Rossby waves.] Such analyses are not available for internal equatorial waves where the introduction of shear leads to a mathematical problem which is nonseparable in its dependence on height and latitude. However, numerical investigations by Lindzen (1970) and Holton (1970) have shown that these waves, too, are absorbed at critical levels.[3] Both of these studies also dealt with the effects of shear zones without critical levels. This case is of importance in relating calculations to observations. It is especially important in the light of the recent observational finding that substantial exchanges of momentum with the mean flow occur in the absence of critical levels (Kousky and Wallace, 1971). Unfortunately, the numerical procedure is cumbersome and time-consuming. Hence, the numerical procedure is ill-suited for comparisons with data. However, the results of Lindzen (1970) and Holton (1970) away from critical levels are so smooth and simple seeming as to suggest the possibility of an analytic asymptotic analysis along the lines of a WKB analysis for ordinary differential equations [see Morse and Feshbach (1953) for example]. The main purpose of this paper is the development of such an analysis for Yanai and Kelvin waves. Not only are the results of interest, but the analysis itself is of interest dealing as it does with a nonseparable partial differential equation with apparent singularities. In the remainder of this paper I will consider the momentum flux due to these waves, and compare the present results with both numerical results and observations. Comparison with observations will give insight into the magnitude of damping in the stratosphere. I will, finally, re-evaluate in the light of our new observational and theoretical knowledge, the relation of these waves to the quasi-biennial oscillation.

2. Equations

In this paper we shall use the identical equations used in Lindzen (1970). We will consider linearized waves in a Boussinesq fluid on an equatorial β plane. Both perturbations and the basic state are taken to be in hydrostatic balance. The basic flow U is taken to be purely zonal, dependent only on height, and in geostrophic balance. The basic state is characterized by a

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[2] The critical level is that level at which the mean horizontal flow Doppler-shifts the wave frequency to zero.

[3] Strictly speaking, Lindzen (1970) only investigated the behavior of the gravest easterly mode (called the Yanai wave) and the gravest westerly mode (called the Kelvin wave). Holton (1970) studied only the Kelvin wave.

constant stability S where

$$S\approx-\frac{\partial}{\partial z}(\ln\bar{\rho}), \tag{1}$$

where $\bar{\rho}$ is the basic density. The Richardson number of the basic state is assumed to be large. The dependence of the perturbations on time t and west-east distance x is taken to be of the form

$$e^{i(kx+\omega t)}, \tag{2}$$

where ω is the wave frequency, and k the zonal wavenumber. The resulting perturbation equations are

$$(i\omega+ikU+\alpha)u+w\frac{dU}{dz}=-ik\Phi+\beta yv, \tag{3}$$

$$(i\omega+ikU+\alpha)v=-\frac{\partial\Phi}{\partial y}-\beta yu, \tag{4}$$

$$\frac{\partial\Phi}{\partial z}=-g\frac{\delta\rho}{\bar{\rho}}, \tag{5}$$

$$(i\omega+ikU+\alpha)\frac{\delta\rho}{\bar{\rho}}-wS+\frac{\beta y}{g}\frac{dU}{dz}v=0 \tag{6}$$

$$iku+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{7}$$

where

$$\Phi\equiv\frac{\delta p}{\bar{\rho}}; \tag{8}$$

α is an arbitrary damping coefficient; u, v, w, δp and $\delta\rho$ are the perturbation zonal, southerly, and vertical velocities, and pressure and density, respectively; y and z the southerly and vertical distances; β is $2\Omega/a$; a and Ω the earth's radius and rotation rate; and overbars refer to the basic state. Taking the Richardson number as

$$\mathrm{Ri}=gS\bigg/\left(\frac{dU}{dz}\right)^2, \tag{9}$$

and dropping terms $O(\mathrm{Ri}^{-1})$ compared to one, we can derive from Eqs. (3)–(7) a single equation for Φ:

$$(\beta^2y^2-\hat{\omega}^2)^2\frac{\partial^2\Phi}{\partial z^2}+2\beta y(\beta^2y^2-\hat{\omega}^2)\frac{dU}{dz}\frac{\partial^2\Phi}{\partial z\partial y}+gS(\beta^2y^2-\hat{\omega}^2)\frac{\partial^2\Phi}{\partial y^2}+\beta\frac{dU}{dz}\left[\frac{\beta y^2}{gS}(\beta^2y^2-\hat{\omega}^2)\frac{d^2U}{dz^2}-(\beta^2y^2+\hat{\omega}^2)\right]\frac{\partial\Phi}{\partial z}$$
$$-2gS\beta^2y\frac{\partial\Phi}{\partial y}+\frac{gSk}{\hat{\omega}}[\beta(\beta^2y^2+\hat{\omega}^2)-k\hat{\omega}(\beta^2y^2-\hat{\omega}^2)]\Phi=0, \tag{10}$$

where

$$\hat{\omega}\equiv\omega+kU-i\alpha.$$

We shall adopt the convention that $\hat{\omega}>0$, in which case westerly waves are associated with negative k, and easterly waves with positive k.

Details may be found in Lindzen (1970). Other fields are related to Φ by the following equations:

$$v=(\beta^2y^2-\hat{\omega}^2)^{-1}\left[ik\beta y\Phi-i\hat{\omega}\left(\frac{\beta y}{gS}\frac{dU}{dz}\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\right)\Phi\right], \tag{11}$$

$$\frac{\delta\rho}{\bar{\rho}}=-\frac{1}{g}\frac{\partial\Phi}{\partial z}, \tag{12}$$

$$w=\frac{1}{gS}\left(-i\hat{\omega}\frac{\partial\Phi}{\partial z}+\beta y\frac{dU}{dz}v\right), \tag{13}$$

$$u=-\frac{1}{\beta y}\left(\frac{\partial\Phi}{\partial y}+i\hat{\omega}v\right). \tag{14}$$

The Boussinesq approximation is by no means necessary, though it helps simplify an already cumbersome problem. Its main effect is to eliminate a factor $\bar{\rho}^{-\frac{1}{2}}$ that would otherwise appear in the solutions for u, v, w, $\delta\rho/\bar{\rho}$ and Φ. The results are otherwise similar provided that S is identified with $(1/\bar{T})[(\partial\bar{T}/\partial z)+(g/c_p)]$ in the atmosphere.

Our boundary conditions on Φ will be $\Phi\rightarrow0$ as $y^2\rightarrow\infty$. We will assume that a wave is forced at $z=0$, and only the upward travelling wave will be considered.

3. Mathematical solution

The problem we are confronted with is obtaining an approximate analytic solution to (10). Numerical solutions for certain cases were obtained in Lindzen (1970). The procedure I shall use is the "two-variable" technique described in Cole (1968). In order to use this technique, the characteristic scale for the variation of $U(z)$ must be much larger than the characteristic vertical scale of the waves we are solving for (essentially a local vertical wavelength divided by 2π). Since the latter is about 1–2 km for observed waves, this is not a particularly severe requirement. The next step is to replace z by "slow" and "fast" height variables (U being taken to be a function solely of the slow variable), where the slow variable is of the form

$$\tau=\epsilon z,\quad \epsilon\ll1, \tag{15}$$

where ϵ is chosen so that the τ derivatives of U are of order unity. Then,

$$\frac{dU}{dz}=\epsilon\frac{dU}{d\tau}=\epsilon U', \tag{16a}$$

$$\frac{d^2U}{dz^2}=\epsilon^2\frac{d^2U}{d\tau^2}=\epsilon^2 U''. \tag{16b}$$

For our "fast" variable we choose

$$\hat{z}=\int_0^z f(\tau)dz, \tag{17}$$

where $f(\tau)$ is determined in the course of solution. In addition, because (10) is nonseparable in its z and y dependence, it proves necessary to replace y by a scaled variable

$$\xi=y/l(\tau), \tag{18}$$

where

$$l=l_0(\tau)+\epsilon l_1(\tau)+\cdots, \tag{19}$$

and where l_0, l_1, ..., are also determined in the course of solution. Now,

$$\Phi=\Phi(\hat{z},\xi,\tau), \tag{20}$$

and

$$\frac{\partial\Phi}{\partial z}=\epsilon\frac{\partial\Phi}{\partial\tau}+f(\tau)\frac{\partial\Phi}{\partial\hat{z}}-\epsilon\frac{\xi}{l}\frac{dl}{d\tau}\frac{\partial\Phi}{\partial\xi}, \tag{20a}$$

$$\frac{\partial\Phi}{\partial y}=\frac{1}{l}\frac{\partial\Phi}{\partial\xi}, \tag{20b}$$

$$\frac{\partial^2\Phi}{\partial y^2}=\frac{1}{l^2}\frac{\partial^2\Phi}{\partial\xi^2}, \tag{20c}$$

$$\frac{\partial^2\Phi}{\partial z\partial y}=\frac{\epsilon}{l}\frac{\partial^2\Phi}{\partial\xi\partial\tau}+\frac{f}{l}\frac{\partial^2\Phi}{\partial\hat{z}\partial\xi}-\epsilon\frac{\xi}{l^2}\frac{dl}{d\tau}\frac{\partial^2\Phi}{\partial\xi^2}-\epsilon\frac{1}{l^2}\frac{dl}{d\tau}\frac{\partial\Phi}{\partial\xi}, \tag{20d}$$

$$\begin{aligned}\frac{\partial^2\Phi}{\partial z^2}=\epsilon\left[\epsilon\frac{\partial^2\Phi}{\partial\tau^2}+f\frac{\partial^2\Phi}{\partial\hat{z}\partial\tau}-\epsilon\frac{\xi}{l}\frac{dl}{d\tau}\frac{\partial^2\Phi}{\partial\xi\partial\tau}\right]+\left[\epsilon\frac{df}{d\tau}\frac{\partial\Phi}{\partial\hat{z}}+\epsilon f\frac{\partial^2\Phi}{\partial\hat{z}\partial\tau}+f^2\frac{\partial^2\Phi}{\partial\hat{z}^2}-\epsilon\frac{\xi}{l}\frac{dl}{\partial\tau}f\frac{\partial^2\Phi}{\partial\hat{z}\partial\xi}\right]\\ -\epsilon\left[\epsilon\xi\frac{d^2\ln l}{d\tau^2}\frac{\partial\Phi}{\partial\xi}+\epsilon\xi\frac{d\ln l}{d\tau}\frac{\partial^2\Phi}{\partial\xi\partial\tau}+\xi\frac{d\ln l}{d\tau}f\frac{\partial^2\Phi}{\partial\xi\partial\hat{z}}-\epsilon\xi\frac{d\ln l}{d\tau}\left(\frac{d\ln l}{d\tau}\frac{\partial\Phi}{\partial\xi}+\xi\frac{d\ln l}{d\tau}\frac{\partial^2\Phi}{\partial\xi^2}\right)\right].\end{aligned} \tag{20e}$$

Substituting Eqs. (20a)–(20e) into (10) we get

$$\begin{aligned}(\beta^2l^2\xi^2-\hat{\omega}^2)^2\left[f^2\frac{\partial^2\Phi}{\partial\hat{z}^2}+\epsilon\left(2f\frac{\partial^2\Phi}{\partial\hat{z}\partial\tau}+\frac{df}{d\tau}\frac{\partial\Phi}{\partial\hat{z}}-2\xi\frac{d\ln l}{d\tau}f\frac{\partial^2\Phi}{\partial\hat{z}\partial\xi}\right)+O(\epsilon^2)\right]+2\beta\xi(\beta^2l^2\xi^2-\hat{\omega}^2)\epsilon U'\left[f\frac{\partial^2\Phi}{\partial\xi\partial\hat{z}}+O(\epsilon)\right]\\ +gS(\beta^2l^2\xi^2-\hat{\omega}^2)\frac{1}{l^2}\frac{\partial^2\Phi}{\partial\xi^2}+\beta\epsilon U'[-(\beta^2l^2\xi^2+\hat{\omega}^2)+O(\epsilon^2)]\left[f\frac{\partial\Phi}{\partial\hat{z}}+O(\epsilon)\right]-2gS\beta^2\xi\frac{\partial\Phi}{\partial\xi}\\ +\frac{gSk}{\hat{\omega}}[\beta(\beta^2l^2\xi^2+\hat{\omega}^2)-k\hat{\omega}(\beta^2l^2\xi^2-\hat{\omega}^2)]\Phi=0.\end{aligned} \tag{21}$$

In order to complete the ordering of (21) in terms of powers of ϵ, we must introduce (19) for l and also assume for Φ

$$\Phi=\Phi_0(\hat{z},\tau,\xi)+\epsilon\Phi_1(\hat{z},\tau,\xi)+\cdots. \tag{22}$$

Then to zeroth order in ϵ (21) becomes

$$\begin{aligned}(\beta^2\xi^2l_0^2-\hat{\omega}^2)f^2\frac{\partial^2\Phi_0}{\partial z^2}\\ +\frac{gS}{l_0^2}(\beta^2\xi^2l_0^2-\hat{\omega}^2)\frac{\partial}{\partial\xi}\left[\frac{1}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}\frac{\partial\Phi_0}{\partial\xi}\right]\\ +\frac{gSk}{\hat{\omega}}[\beta(\beta^2\xi^2l_0^2+\hat{\omega}^2)-k\hat{\omega}(\beta^2\xi^2l_0^2-\hat{\omega}^2)]\Phi_0=0.\end{aligned} \tag{23}$$

Eq. (23) involves τ only parametrically, and its solutions are simply those given in Lindzen (1967), Lindzen and Matsuno (1968) and Holton and Lindzen (1968).

There are a countably infinite set of these solutions designated by $n=-1, 0, 1, 2, \ldots$. For each of the solutions, the scaling factors, f and l_0, turn out to be related simply as

$$\beta^2 l_0^4=\frac{gS}{f^2}=gh^{(n)}, \tag{24}$$

where $h^{(n)}$, the equivalent depth of tidal theory (*viz.* Chapman and Lindzen, 1970), is determined for each n. *This means that* $\hat{z}$ *and* ξ *represent different scalings for each* n. Once $\Phi_0^{(n)}$ is obtained, the remaining fields to zeroth order in ϵ are given by [see Eqs. (11)–(14)]

$$v_0=(\beta^2 l_0^2\xi^2-\hat{\omega}^2)^{-1}\left(ik\beta l_0\xi\Phi_0-i\hat{\omega}\frac{1}{l_0}\frac{\partial\Phi_0}{\partial\xi}\right), \tag{25}$$

$$w_0=\frac{1}{gS}\hat{\omega}f\Phi_0, \tag{26}$$

$$u_0=-\frac{1}{\beta l_0\xi}\left(\frac{1}{l_0}\frac{\partial\Phi_0}{\partial\xi}+i\hat{\omega}v_0\right), \tag{27}$$

$$\frac{\delta\rho_0}{\bar{\rho}}=-\frac{i}{g}f\Phi_0. \tag{28}$$

For $n=-1$, we have the Kelvin wave discussed by Holton and Lindzen (1968) for which

$$\Phi_0^{-1}=A_0^{-1}(\tau)\exp(-\xi^2/2)\exp(i\hat{z}), \tag{29}$$

$$\sqrt{gh^{(-1)}}=-\frac{\hat{\omega}}{k}. \tag{30}$$

Since the Kelvin wave is westerly, k is negative. In general, $A_0^{(n)}$ will be determined at first order in ϵ (as will l_1). For $n=0$ we have the Yanai wave discussed in Lindzen and Matsuno (1968) for which

$$\Phi_0^{(0)}=A_0^{(0)}(\tau)[2\xi\exp(-\xi^2/2)]\exp(i\hat{z}), \tag{31}$$

$$\sqrt{gh^{(0)}}=\frac{a\hat{\omega}^2}{\Omega}\Big/\left(2-\frac{s\hat{\omega}}{\Omega}\right), \tag{32}$$

where $s=ak$, corresponding to the zonal wavenumber at the equator (i.e., the number of waves around the equatorial latitude belt). Since this wave is easterly, k is positive.

For $n\geqslant 1$, the solutions are discussed in Lindzen (1967) and Lindzen and Matsuno (1968), and given by

$$\Phi_0^{(n)}=A_0^{(n)}(\tau)[-2nq(\tau)H_{n-1}(\xi)+H_{n+1}(\xi)]\times\exp(-\xi^2/2)\exp(i\hat{z}), \tag{33}$$

$$\sqrt{gh^{(n)}}=\frac{a\Omega(2n+1)}{s^2\left[\left(\frac{2\Omega}{s\hat{\omega}}\right)-1\right]}\times\left\{1\pm\left[1-\left(\frac{\hat{\omega}}{\Omega}\right)^2\left(\frac{s}{2n+1}\right)^2\left(\frac{2\Omega}{s\hat{\omega}}-1\right)\right]^{\frac{1}{2}}\right\}, \tag{34}$$

$$q(\tau)=\frac{1+\left(\frac{k}{\hat{\omega}}\right)\sqrt{gh^{(n)}}}{1-\left(\frac{k}{\hat{\omega}}\right)\sqrt{gh^{(n)}}}. \tag{35}$$

In addition to the solutions given by (29), (31) and (33), there are also solutions proportional to e^{-iz}; however, these solutions represent downward propagating wave energy.

For $n\geqslant 1$, since $q(\tau)\neq 0$, our solutions are no longer simply factorizable into functions of τ, $\hat{z}$ and ξ alone. This factorizability is essential to the normal application of "two-variable" techniques. Therefore, for $n\geqslant 1$ a somewhat new approach is needed wherein q is expanded in powers of ϵ. The calculations for $n\geqslant 1$ are not yet complete, and I will restrict myself to the Kelvin and Yanai waves ($n=-1$, 0) in this paper. Since these are the only waves that have been positively identified in the tropical stratosphere, the results should still be of practical importance.

For $n=-1$ and 0, the $O(\epsilon)$ part of Eq. (21) is

$$\frac{f^2l_0^2}{gS}\frac{\partial^2\Phi_1}{\partial\hat{z}^2}+\frac{\partial}{\partial\xi}\left[\frac{1}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}\frac{\partial\Phi_1}{\partial\xi}\right]+\frac{l_0^2k}{\hat{\omega}}\left[\beta\frac{(\beta^2\xi^2l_0^2+\hat{\omega}^2)}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}-\frac{k\hat{\omega}}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}\right]\Phi_1$$
$$=-\frac{l_0^2}{gS}\left[I_1+\frac{I_2}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}+\frac{I_3}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)^2}\right], \tag{36}$$

where

$$I_1=2f\frac{\partial^2\Phi_0}{\partial\hat{z}\partial\tau}+\frac{df}{d\tau}\frac{\partial\Phi_0}{\partial\hat{z}}-2\xi\frac{f}{l_0}\frac{dl_0}{d\tau}\frac{\partial^2\Phi_0}{\partial\hat{z}\partial\xi}, \tag{37}$$

$$I_2=4l_0l_1\beta^2\xi^2f^2\frac{\partial^2\Phi_0}{\partial\hat{z}^2}+2\beta\xi U'f\frac{\partial^2\Phi_0}{\partial\hat{z}\partial\xi}-2gS\frac{l_1}{l_0^3}\frac{\partial^2\Phi_0}{\partial\xi^2}, \tag{38}$$

$$I_3=2gS\beta^2\xi^2\frac{l_1}{l_0}\frac{\partial^2\Phi_0}{\partial\xi^2}-\beta U'f(\beta^2\xi^2l_0^2+\hat{\omega}^2)\frac{\partial\Phi_0}{\partial\hat{z}}+\frac{gSk}{\hat{\omega}}2l_0l_1\beta^2\xi^2(\beta-k\hat{\omega})\Phi_0. \tag{39}$$

The left-hand side of (36) is exactly the same as the equation for Φ_0 [i.e., Eq. (23)]. Hence any part of the right-hand side of (36) which is proportional to Φ_0 (or more precisely, has a projection on Φ_0) will produce a spurious resonant response in Φ_1. The general two-variable approach is to choose A_0 to suppress any part of the inhomogeneity proportional to Φ_0. In the present problem there is an additional complication; two of the terms in the inhomogeneity are singular where $\beta^2\xi^2 l_0^2-\hat{\omega}^2=0$ and represent infinite forcings there. Now the projection of the inhomogeneity on any zeroth-order solution is determined by integrating the product of that solution and the inhomogeneity over all ξ's. If you use the Cauchy principle values of the integrals, then the projections of the term $I_2/(\beta^2\xi^2l_0^2-\hat{\omega}^2)$ will be finite. However, the projections of $I_3/(\beta^2\xi^2l_0^2-\hat{\omega}^2)^2$ will still be infinite. Therefore, it is necessary (and possible) to choose l_1 so that I_3 is proportional to $(\beta^2\xi^2l_0^2-\hat{\omega}^2)$, in which case principle values will remain finite. As I show in the Appendix, this procedure leads to the following solution for l_1 for both $n=-1$ and $n=0$:

$$l_1=-\tfrac{1}{2}i\frac{U'}{\sqrt{gS}}l_0. \tag{40}$$

In addition, as shown in the Appendix, it is fortunately the case for the above choice of l_1 that

$$\frac{I_2}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}+\frac{I_3}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)^2}=0. \tag{41}$$

Thus, the determination of A_0 requires only that I_1 have no projection on Φ_0.

Now,

$$\begin{aligned}\begin{Bmatrix} I_1^{(-1)} \\ I_1^{(0)}\end{Bmatrix} &=2if\begin{Bmatrix}\dfrac{dA_0^{(-1)}}{d\tau}\exp(-\xi^2/2) \\ 2\dfrac{dA_0^{(0)}}{d\tau}\xi\exp(-\xi^2/2)\end{Bmatrix}\exp(i\hat{z}) \\ &+i\frac{df}{d\tau}\begin{Bmatrix}A_0^{(-1)}\exp(-\xi^2/2) \\ 2A_0^{(0)}\xi\exp(-\xi^2/2)\end{Bmatrix}\exp(i\hat{z}) \\ &-2i\frac{f}{l_0}\frac{dl_0}{d\tau}\xi\frac{\partial}{\partial\xi}\begin{Bmatrix}\Phi_0^{(-1)} \\ \Phi_0^{(0)}\end{Bmatrix}.\end{aligned} \tag{42}$$

The first two terms in (42) are proportional to $\Phi_0^{(n)}$. The projection of the third term is given by

$$\begin{aligned}&-2i\frac{f}{l_0}\frac{dl_0}{d\tau}\frac{\displaystyle\int_{-\infty}^{\infty}\Phi_0\xi\frac{\partial\Phi_0}{\partial\xi}d\xi}{\displaystyle\int_{-\infty}^{\infty}\Phi_0^2d\xi}\begin{Bmatrix}A_0^{(-1)}\exp(-\xi^2/2) \\ 2A_0^{(0)}\xi\exp(-\xi^2/2)\end{Bmatrix}\exp(i\hat{z}) \\ &=-i\frac{f}{l_0}\frac{dl_0}{d\tau}\begin{Bmatrix}A_0^{(-1)}\exp(-\xi^2/2) \\ 2A_0^{(0)}\xi\exp(-\xi^2/2)\end{Bmatrix}\exp(i\hat{z}),\end{aligned} \tag{43}$$

where the integration has been performed by integrating by parts.

Thus, the projection of I_1 on Φ_0 is proportional to

$$P=2f\frac{dA_0}{d\tau}+\frac{df}{d\tau}A_0+\frac{f}{l_0}\frac{dl_0}{d\tau}A_0, \tag{44}$$

where $P=0$ implies

$$\frac{1}{A_0}\frac{dA_0}{d\tau}=-\frac{1}{2}\left(\frac{1}{f}\frac{df}{d\tau}+\frac{1}{l_0}\frac{dl_0}{d\tau}\right). \tag{45}$$

Eq. (45) is immediately integrable, yielding

$$A_0=\frac{\text{constant}}{\sqrt{fl_0}}. \tag{46}$$

The interpretation of A_0 is quite simple. The $1/\sqrt{f}$ dependence is what one normally gets in one-dimensional WKB analyses. The present waves, however, change in horizontal extent l_0 with height. The dependence represents the change of amplitude needed because the total wave energy is being confined to narrower or wider regions.

With (46), our solution to zeroth order in ϵ for $n=-1$, 0 is complete. The remaining inhomogeneity in I_1 forces an $O(\epsilon)$ correction. However, the evaluation of this correction requires that we know zeroth-order solutions for $n\geqslant 1$. We must, therefore, content ourselves for the moment with the zeroth-order solutions for Kelvin and Yanai waves.

4. Review of zeroth-order solutions

In this section I bring together the results for Kelvin and Yanai waves as derived above, together with solutions for other fields as derived from (25)–(28).

a. Kelvin wave

Let

$$\frac{\hat{\omega}}{k}\equiv c-U, \tag{47}$$

where c is the zonal phase speed relative to the ground. Then

$$\Phi_0=\text{constant}\times(c-U)^{\frac{1}{2}}\exp(-\xi^2/2)\exp(i\hat{z}), \tag{48}$$

$$v_0=0, \tag{49}$$

$$w_0=-\text{constant}\times\frac{k}{\sqrt{gS}}(c-U)^{\frac{1}{2}}\exp(-\xi^2/2)\exp(i\hat{z}), \tag{50}$$

$$u_0=\text{constant}\times(c-U)^{-\frac{1}{2}}\exp(-\xi^2/2)\exp(i\hat{z}), \tag{51}$$

$$\frac{\delta\rho_0}{\bar{\rho}}=-i\ \text{constant}\times\frac{\sqrt{gS}}{g}(c-U)^{-\frac{3}{2}}\exp(-\xi^2/2)\times\exp(i\hat{z}), \tag{52}$$

where

$$\hat{z}=\int_0^z f dz, \tag{53}$$

$$f=\sqrt{gS}/(c-U), \tag{54}$$

$$\xi=y/l, \tag{55}$$

$$l\approx l_0\left(1-\frac{1}{2}\frac{\frac{dU}{dz}}{\sqrt{gS}}i\right). \tag{56}$$

Or using (9),

$$l=l_0(1-\tfrac{1}{2}\,\mathrm{Ri}^{-\frac{1}{2}}i), \tag{57}$$

where

$$l_0=[(c-U)/\beta]^{\frac{1}{2}}. \tag{58}$$

Using (57),

$$\exp(-\xi^2/2)\approx\exp\left[-\frac{1}{2}\frac{y^2}{l_0^2}(1+\mathrm{Ri}^{-\frac{1}{2}}i)\right]. \tag{59}$$

b. Yanai wave

$$\Phi_0=\text{constant}\times(U-c)^{\frac{1}{2}}\left[1-\frac{k^2}{\beta}(U-c)\right]^{-\frac{1}{4}}\xi \times\exp(-\xi^2/2)\exp(i\hat{z}), \tag{60}$$

$$v_0=i\,\text{constant}\times\beta k^{-2}(U-c)^{-\frac{3}{2}}\left[1-\frac{k^2}{\beta}(U-c)\right]^{\frac{1}{4}} \times\exp(-\xi^2/2)\exp(i\hat{z}), \tag{61}$$

$$w_0=\text{constant}\times\frac{\beta k^{-1}}{\sqrt{gS}}(U-c)^{-\frac{1}{2}}\left[1-\frac{k^2}{\beta}(U-c)\right]^{\frac{3}{4}}\xi \times\exp(-\xi^2/2)\exp(i\hat{z}), \tag{62}$$

$$u_0=\text{constant}\times\beta k^{-2}(U-c)^{-\frac{3}{2}}\left[1-\frac{k^2}{\beta}(U-c)\right]^{\frac{3}{4}}\xi \times\exp(-\xi^2/2)\exp(i\hat{z}), \tag{63}$$

$$\frac{\delta\rho_0}{\bar{\rho}}=-i\,\text{constant}\times\frac{\beta k^{-2}}{g}\sqrt{gS}(U-c)^{-\frac{3}{2}} \times\left[1-\frac{k^2}{\beta}(U-c)\right]^{\frac{3}{4}}\xi\exp(-\xi^2/2)\exp(i\hat{z}), \tag{64}$$

where again (53), (55), (56), (57) and (59) hold. However, now

$$f=\sqrt{gS}\frac{\beta}{k^2}(U-c)^{-2}\left[1-\frac{k^2}{\beta}(U-c)\right] \tag{65}$$

and

$$l_0=\frac{k}{\beta}(U-c)\left[1-\frac{k^2}{\beta}(U-c)\right]^{-\frac{1}{2}}. \tag{66}$$

c. General comments

The above equations [(48)–(66)] describe waves which are changing both their vertical wavenumber f and horizontal extent l_0, as they propagate upward through a medium with varying zonal current U. At the equator, the magnitude of the horizontal velocity oscillation associated with the wave increases as $|U-c|$ decreases, and much more rapidly for the Yanai wave than for the Kelvin wave.

The only effect of the terms in (10) proportional to dU/dz on Φ_0 is to introduce an $O(\epsilon)$ correction to the horizontal scaling l. This correction is imaginary and, as a result, the lines of constant phase are no longer horizontal lines, but "parabolas" defined by

$$\hat{z}-\tfrac{1}{2}\,\mathrm{Ri}^{-\frac{1}{2}}\xi^2=\text{constant}. \tag{67}$$

The bending of the phase lines corresponds to the focusing (or defocusing) of the wave energy as the wave propagates into a region where it is narrower (or wider). When dU/dz terms are neglected in (10) as is the case in Holton (1971), then this effect is absent since the wave at any given level no longer knows that is is moving into a region where its horizontal extent will be different.

d. Introduction of damping

Our equations are easily evaluated for any choices of S, k and ω. It should, therefore, be possible to compare our results with the numerical results obtained by Lindzen (1970) and Holton (1970). However, there are two significant differences between these results and those in this paper. Holton did not use the Boussinesq approximation, and both Lindzen and Holton used finite damping α. The former can be largely corrected by multiplying the relevant fields in the present study by $e^{x/2}$, where x is the height in scale heights. As concerns the latter, it should be pointed out that although our discussion has generally referred to $\hat{\omega}$ (and hence c) as a real quantity, there is no such restriction implicit in the analysis. Hence, the effects of damping may be evaluated simply by adding an imaginary part, $i\alpha$, to $\hat{\omega}$ (see p. 610). In most of our explicit calculations, α will only be a small perturbation to ω, and its effects will be small except insofar as it introduces an imaginary part to f; this will give rise to exponential decay of amplitude with height. Moreover, for small α we can use simplified perturbation expressions for f.

For the Kelvin wave the perturbed f is

$$f^{(-1)}\approx\frac{\sqrt{gS}}{c-U}\left[1-\frac{i\alpha}{k(c-U)}\right], \tag{68}$$

and for the Yanai wave

$$f^{(0)}\approx\sqrt{gS}\frac{\beta}{k^2}\frac{1}{(U-c)^2}\left[1-\frac{k^2}{\beta}(U-c)\right]$$

$$\times\left\{1+\frac{i\alpha}{k(U-c)}\left[\frac{2-\frac{k^2}{\beta}(U-c)}{1-\frac{k^2}{\beta}(U-c)}\right]\right\}. \quad (69)$$

Small α means

$$\alpha\ll k(c-U) \text{ for the Kelvin wave,}$$

and

$$\alpha\ll k(U-c)\left[1-\frac{k^2}{\beta}(U-c)\right]$$

for the Yanai wave.

In the remainder of this paper I will restrict myself to the approximate expressions (68) and (69) when evaluating the effects of dissipation.

Before comparing my results with either numerical calculations or observations, I will discuss the vertical momentum fluxes due to Kelvin and Yanai waves.

5. Vertical fluxes of zonal momentum

As shown by Bretherton (1969), the vertical flux of zonal momentum due to linearized waves in a rotating fluid is

$$F_m=\bar{p}\overline{uw}-\bar{p}f\overline{\eta w}, \quad (70)$$

where the overbar refers to a time (or longitude) average,

$$\eta=\frac{iv}{\hat{\omega}} \quad (71)$$

is the southerly displacement associated with the wave, and f the local Coriolis parameter. For a plane rotating fluid where f is constant, F_m is independent of height provided that $U-c\neq0$ and there is no damping. In the present case, however, where wave shape is changing with height we no longer expect F_m to be locally independent of height, but $\langle F_m\rangle$, where $\langle\ \rangle$ refers to the integral from $y=-\infty$ to $y=+\infty$, does prove height-independent (providing $U-c\neq0$ and $\alpha=0$), at least to zeroth order in ϵ. Since we are dealing with a Boussinesq fluid we will consider $\tilde{F}_m=F_m/\bar{p}$ rather than $\tilde{F}_m$.

Then, using Eqs. (49), (50), (51) and (71), we find for the Kelvin wave that

$$\langle\tilde{F}_m\rangle\approx\langle\overline{u_0w_0}\rangle$$

$$=\tfrac{1}{2}\,\text{constant}^2\frac{(-k)}{\sqrt{gS}}\frac{1}{\sqrt{\beta}}\int_{-\infty}^{\infty}\exp(-\xi^2)d\xi. \quad (72)$$

Recalling that

$$dy\approx l_0 d\xi, \quad (73)$$

and that k is negative, we have

$$\int_{-\infty}^{\infty}\exp(-\xi^2)d\xi=\sqrt{\pi}. \quad (74)$$

Clearly, $\langle\tilde{F}_m\rangle$ is independent of height for the Kelvin wave in the absence of critical levels and damping.

Using (62) and (63) we find for the Yanai wave that

$$\langle\overline{u_0w_0}\rangle=\tfrac{1}{2}\,\text{constant}^2\frac{\beta/k^2}{\sqrt{gS}}(U-c)^{-1}\left[1-\frac{k^2}{\beta}(U-c)\right]$$

$$\times\int_{-\infty}^{\infty}\xi^2\exp(-\xi^2)d\xi, \quad (75)$$

and using (55), (61), (62) and (71)

$$f\langle\overline{\eta_0w_0}\rangle=\tfrac{1}{2}\,\text{constant}^2\frac{\beta/k^2}{\sqrt{gS}}(U-c)^{-1}$$

$$\times\int_{-\infty}^{\infty}\xi^2\exp(-\xi^2)d\xi, \quad (76)$$

where

$$\int_{-\infty}^{\infty}\xi^2\exp(-\xi^2)d\xi=\frac{\sqrt{\pi}}{2}. \quad (77)$$

Neither $\langle\overline{u_0w_0}\rangle$ nor $f\langle\overline{\eta_0w_0}\rangle$ are independent of height and, indeed, they appear to blow up as $U-c\rightarrow0$. However,

$$\langle\tilde{F}_m\rangle=\langle\overline{u_0w_0}\rangle-f\langle\overline{\eta_0w_0}\rangle,$$

$$=-\tfrac{1}{2}\,\text{constant}^2\frac{1}{\sqrt{gS}}\frac{\sqrt{\pi}}{2}, \quad (78)$$

is independent of height.

In addition, it is easily shown [using (48), (50), (60) and (62)] that the quantity

$$\frac{\langle\overline{w_0\Phi_0}\rangle}{\hat{\omega}}, \quad (79)$$

which is analogous to Bretherton and Garret's (1969) wave action, is independent of height for both Kelvin and Yanai waves.

Note, that $\langle\tilde{F}_m\rangle$ is positive for westerly Kelvin waves and negative for easterly Yanai waves.

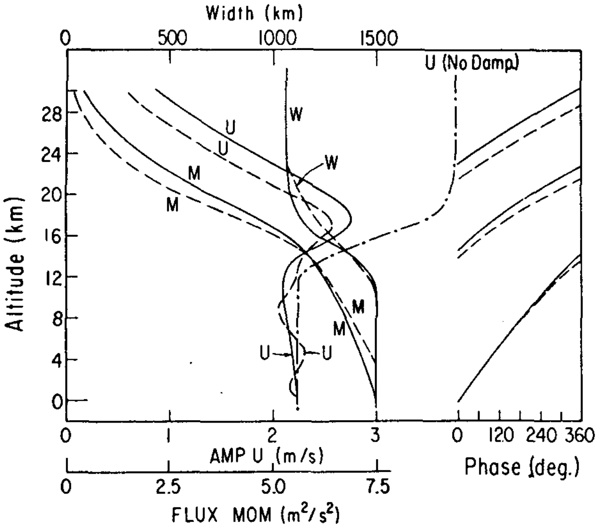

FIG. 1. A comparison of numerical and asymptotic results for the distribution with height of the amplitude of the zonal wind oscillation at the equator (U), the integrated vertical flux of zonal momentum (M), the characteristic north-south half-width of the wave (W), and the phase of the zonal wind oscillation for an internal equatorial Kelvin wave without a critical layer. Solid lines, asymptotic calculation with damping; dashed lines, numerical calculation with damping; dashed-dotted lines, asymptotic calculation without damping.

In the presence of small damping both $\langle \widetilde{F}_m \rangle$ and wave action will decay as

$$\exp\left[-2\int_0^z \mathrm{Im}(f)dz\right], \tag{80}$$

where the imaginary part of f is given approximately by (68) for the Kelvin wave and (69) for the Yanai wave.

6. Comparison with numerical integrations

The comparison of the present results with those obtained earlier by numerical integration (Lindzen, 1970) is of great interest. The numerical calculations were tedious and very lengthy; use of the present formulas, provided they are sufficiently accurate, would represent a substantial gain in flexibility and economy.

In Lindzen (1970) mean flows of the following form were considered:

$$U = U_0 \tanh\left(\frac{z-15\text{ km}}{L}\right). \tag{81}$$

Kelvin and Yanai waves in flows with and without critical levels were studied. *A priori*, the present analysis is inadequate at critical levels; therefore, for the moment, we will consider the examples without critical levels. The following choice of parameters was made in Lindzen (1970) where the reasoning behind the choices is discussed. For the Kelvin wave we used

$$U_0 = 0.25\times 51.6 \text{ m sec}^{-1}, \tag{82}$$

$$\omega = 0.75\times 0.3318\Omega, \tag{83}$$

$$k = -3/(6400 \text{ km}), \tag{84}$$

while for the Yanai wave, we had

$$U_0 = -0.25\times 51.7 \text{ m sec}^{-1}, \tag{85}$$

$$\omega = 0.75\times 0.3328\Omega, \tag{86}$$

$$k = 3/(6400 \text{ km}). \tag{87}$$

For each of the two waves, we used

$$L = 2.5 \text{ km} \tag{88}$$

$$\alpha = (1/40 \text{ days}) \exp(z/15 \text{ km}). \tag{89}$$

We have evaluated our formulas for the above choices of parameters, and we have also considered the case $\alpha = 0$.

In order to compare numerical and asymptotic solutions we have characterized Kelvin waves by 1) amplitude (of u oscillation) at the equator, 2) latitudinal extent of solution (i.e., distance from equator where amplitude is $e^{-\frac{1}{2}}$ times the amplitude at the equator), 3) total vertical flux of zonal momentum, and 4) phase (in degrees) at the equator. For the Yanai wave, 1) is replaced by maximum amplitude at a fixed level, and 2) by the distance from the equator to the maximum amplitude.

The above characterizations display the major features of the solutions without focussing on the plethora of small-amplitude detail in the numerical solutions.

In Fig. 1 we see numerical and asymptotic results for the Kelvin wave, as well as the vertical distribution of amplitude in the absence of damping. Results for the Yanai wave are shown in Fig. 2. In general, numerical and asymptotic solutions are in good qualitative and quantitative agreement; certainly, no current observations could distinguish between the two. Where significant deviations occur (as for width and phase above 24 km for the Yanai wave), amplitudes decayed to undetectable levels. Notice that maximum amplitudes are 30–50% less with damping than without damping.

The above success with our asymptotic formulas suggests the possibility of using them when critical levels are present. This should be especially likely for the Yanai wave which, because of damping, is attenuated to a large extent well below the critical level. In Lindzen (1970) the following parameters were chosen for the Kelvin wave with a critical level:

$$U_0 = 51.6 \text{ m sec}^{-1}, \tag{90}$$

$$\omega = 0, \tag{91}$$

$$k = -3/(6400 \text{ km}), \tag{92}$$

while for the Yanai wave, we chose

$$U_0 = -51.7 \text{ m sec}^{-1}, \quad (93)$$

$$\omega = 0, \quad (94)$$

$$k = 3/(6400 \text{ km}). \quad (95)$$

For each of the two waves, we used

$$L = 5 \text{ km}, \quad (96)$$

$$\alpha = 1/6 \text{ day}. \quad (97)$$

Numerical and asymptotic results are shown in Figs. 3a and 3b for the Yanai and Kelvin waves, respectively. Results are acceptably close for the Yanai; for the Kelvin wave, agreement is close to within 2.5 km of the critical level (by which point two-thirds of the momentum flux at $z=0$ has been absorbed). Above $z=12.5$ km, the amplitude of u decays for the numerical calculation, while it grows an additional 10% for the asymptotic calculation, it reaches a maximum at 14 km and decays sharply above $z=14$ km.

In summary, our asymptotic results match numerical results with reasonable accuracy away from critical levels. When sufficient damping is present, the asymptotic results also adequately describe waves when critical levels are present.

7. Comparison with observations

As mentioned in the Introduction, the waves we have been discussing have been observed, as best we can tell, in the equatorial lower stratosphere. Yanai waves are generally observed when the prevailing flow is westerly with the easterly phase of the quasi-biennial cycle descending from above, while Kelvin waves are observed when the prevailing flow is easterly and the westerly phase of the quasi-biennial cycle is descending. When we say that such waves are observed, what we mean is that structures are observed with spatial distributions and dispersive properties that are reasonably close to the simple waves described theoretically by Lindzen and Matsuno (1968) and Holton and Lindzen (1968). However, close agreement between theory and observation cannot be expected at this stage. The atmosphere is unlikely to have the pure, monochromatic waves described by the theory. Also, present observations lack the resolution to describe the vertical structures predicted by the theory. Nevertheless, comparisons of theory and observation can lend insight into what is really happening in the atmosphere.

a. Yanai wave

An extensive number of papers by members of the Geophysical Institute of Tokyo University have dealt with the disturbance of the equatorial stratosphere which I have referred to as the Yanai wave. These papers have recently been compiled in a single volume (Syono and Yanai, 1970). For our present purposes the most useful paper is by Maruyama (1967). In the data he analyzed, a Yanai wave was evident with a zonal phase speed

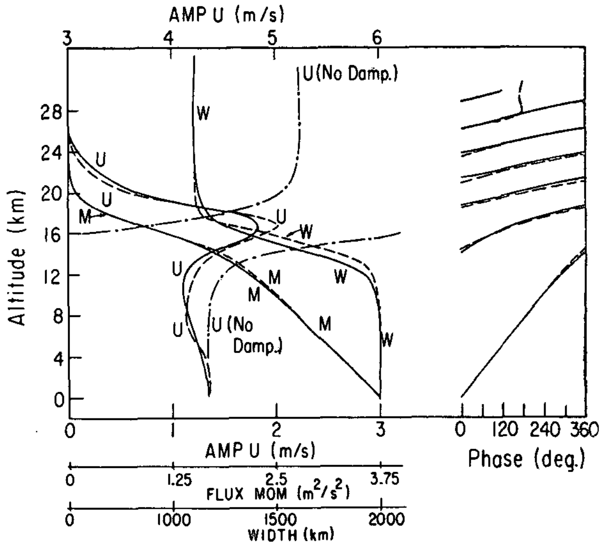

FIG. 2. Same as Fig. 1 except for an internal Yanai wave without a critical level. Also, U now refers to the maximum zonal wind oscillation amplitude at a given altitude, and W to the distance of the maximum from the equator. The upper scale applies to the upper branch of the curve of U for the case without damping.

$$c = -\omega/k = -23 \text{ m sec}^{-1}, \quad (98)$$

and a zonal wavenumber

$$k = 0.63 \times 10^{-3} \text{ km}^{-1}. \quad (99)$$

The Yanai wave's contribution to zonal velocity and temperature are antisymmetric about the equator. Hence, the observed existence of oscillations in zonal wind and temperature at the equator implies that Maruyama's data includes waves other than the Yanai wave. However, if we concentrate on the southerly wind component at the equator we are likely to be looking at what is primarily a Yanai wave. Kapingamarangi (01°02′N, 154°46′E) is sufficiently close to the equator for this purpose. What I propose to do is to take the distribution of mean zonal wind at the equator as reported by Maruyama for the period during which the waves were observed and calculate the behavior of a Yanai wave for which (98) and (99) hold and for various choices of a constant damping coefficient α, comparing the results for southerly wind amplitude at the equator with observations at Kapingamarangi. The mean zonal wind shown by Maruyama is accurately described by the following formula in the region 15–26 km over the equator:

$$U = 3.9 \text{ m sec}^{-1} \text{ km}^{-2} (z-15 \text{ km})^2 \times\left(1-\frac{(z-15 \text{ km})}{7.5 \text{ km}}\right) e^{-z/(4.55\text{km})}, \quad (100)$$

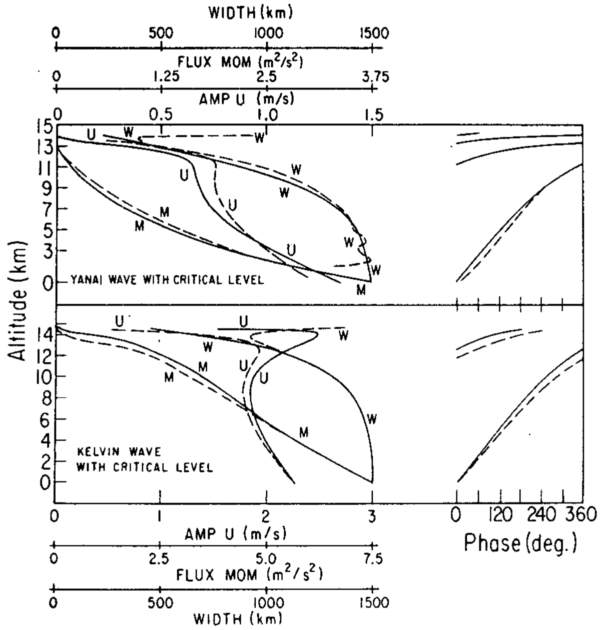

FIG. 3. Same as Figs. 1 and 2 except for Yanai and Kelvin waves with critical levels. Cases without damping not included.

where z is the altitude (km). The distribution is shown graphically in Fig. 4. In calculating wave fields I take the southerly velocity oscillation at 16.5 km to be equal in amplitude to the observed oscillation at 100 mb. Table 1 shows Maryuyama's measured amplitudes for the southerly velocity oscillation at various heights and the standard deviations; also shown are theoretical amplitudes for different choices of α. We see that our calculated amplitude at 21 km is not sensitive to our choice of α; however, the calculated amplitude at 25 km (which is only about 1.5 km below a critical level) is very sensitive to α. In fact our calculations are compatible with observations only for α^{-1} between 14 and 19 days. This is especially significant since the time scale for radiative cooling is between two and three weeks (Lindzen and Goody, 1965), and radiative cooling is the largest known source of damping in the stratosphere.

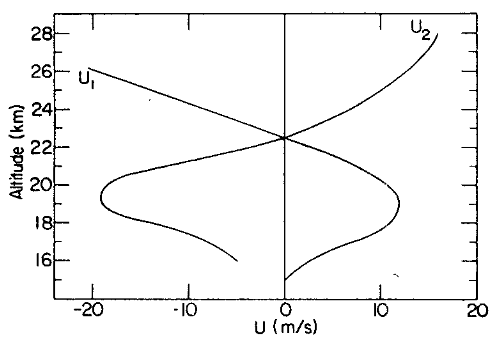

FIG. 4. Distributions of mean zonal wind at the equator. The U_1 curve represents the case where easterlies are descending and Yanai waves are observed; U_2 the case where westerlies are descending and Kelvin waves are observed.

The present results are entirely compatible with the estimated radiative time scales and with the possibility that radiation is the main source for damping.[4]

The sensitivity of wave amplitude at 25 km to our choice of α is, as already mentioned, due to the proximity of this level to the critical level. As we have seen in Section 6 this is a region of rapid variation in wave amplitude and phase. This is seen more clearly in Fig. 5 where the distribution with height of the wave's meridional velocity amplitude at the equator, integrated zonal momentum flux, latitudinal scale, and phase are shown for $\alpha^{-1}=14$ days and $g^{-1}=17.5$ days. Also shown are the levels at which radiosonde data were available. Clearly, these were insufficient to resolve the calculated phase and amplitude structure. Indeed, the data might easily be misinterpreted as showing little or no phase variation between 22.5 and 25 km, even if the distributions shown in Fig. 5 were correct. Finally, we should note for the values of α used that zonal momentum should be fairly uniformly deposited in the mean flow between 16.5 and 25 km.

TABLE 1. Observed and theoretical values for the amplitude of the meridional velocity component of a Yanai wave at selected levels over the equator. Theoretical values are shown for different choices of the dissipative time scale.

	Altitude		
	100 mb (16.5 km)	50 mb (21 km)	25 mb (25 km)
	Observed amplitudes* (m sec^{-1})		
	3.0±1	2.0±0.9	2.5±1.2
	Theoretical amplitudes (m sec^{-1})		
$\alpha=1/7$ days	3.0	1.91	0.045
$\alpha=1/10.5$ days	3.0	2.37	0.43
$\alpha=1/14$ days	3.0	2.64	1.32
$\alpha=1/17.5$ days	3.0	2.82	2.60
$\alpha=1/21$ days	3.0	2.94	4.08
$\alpha=1/24.5$ days	3.0	3.04	5.64
$\alpha=1/28$ days	3.0	3.11	7.17
$\alpha=0$	3.0	3.65	38.8

* For Kapingamarangi (IN), from Maruyama (1967).

b. Kelvin wave

The Kelvin wave in the lower stratosphere has been described in Wallace and Kousky (1968a, b) and Kousky and Wallace (1971). A wave was called a Kelvin wave by these authors provided that its phase speed was westerly relative to the mean flow, it was symmetric about the equator, its amplitude decayed away from the equator, and it had no discernible southerly velocity component. Data for two periods (4 July–15 October 1963 and 10 January–23 April 1966) were analyzed in

[4] If there is only radiative damping, then within the Boussinesq approximation, damping will occur in Eq. (6) but not in Eqs. (3) and (4). However, it may be inferred from Lindzen (1968) that this does not significantly alter the situation.

each of which the lower stratosphere had prevailing easterlies with westerlies descending from above. The distribution of U with height near the equator for the first period is shown in Fig. 4. Spectral analyses indicated a peak corresponding to a period of $\sim$12 days with significant energy between 8 and 20 days. Subsequent analyses filtered the data so as to suppress periods outside this range. Some of the results in Kousky and Wallace (1971) are reproduced in Figs. 6 and 7. In Fig. 6 we see a meridional cross section of power density in the frequency range 0.05–0.125 cycle day^{-1}. Two distinct maxima are seen, one near 25 km, the other near 19 km. It should be pointed out that the minimum power density corresponds to amplitudes of $\sim$7.5 m sec^{-1} while maximum densities correspond to amplitudes of $\sim$10 and 11 m sec^{-1}; uncertainty in amplitude at any given level must be at least 1 m sec^{-1}. Fig. 6 also shows that the disturbance width at 19 km is substantially greater than at 25 km, although the substantial spread in latitude between observing stations renders this point somewhat uncertain. Fig. 7 shows the variation of disturbance phase with height based on data from 100, 80, 70, 60, 50, 40, 30, 25, 20 and 15 mb. Phases above and below 50 mb showed no consistent correlation. In both the upper and lower regimes there was downward propagation of phase associated with an estimated vertical wavelength of 7 km in the lower regime and 35 km in the upper regime. Kousky and Wallace estimate a zonal wavenumber between 1 and 2 in the lower regime and a wavenumber equal to 1 in the upper regime.

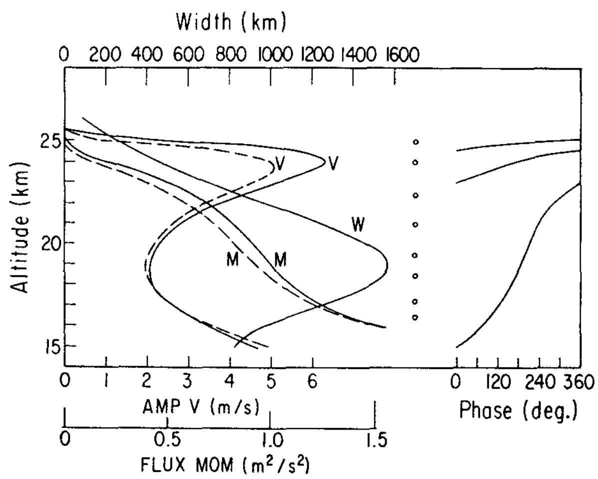

FIG. 5. The distribution with height of the amplitude of the meridional wind oscillation at the equator (V), the integrated vertical flux of zonal momentum (M), the latitudinal scale as measured from the equator (W), and the phase for a Yanai wave propagating through a mean wind field given by U_1 in Fig. 4. Calculations are shown for two choices of α: 1/14 days, dashed lines and 1/17.5 days, (solid lines). Also shown (by open circles) are the approximate levels from which Maruyama (1967) had radiosonde data.

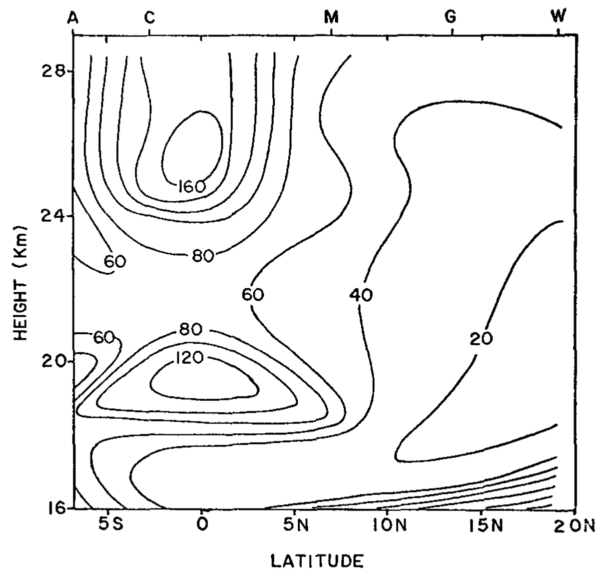

FIG. 6. Meridional cross section of observed integrated power in zonal wind between 0.05 and 0.125 cycle day^{-1} when Kelvin waves are observed. Numbers must be multiplied by $\pi/4$ to give measures of the square of the characteristic zonal wind oscillation. From Kousky and Wallace (1971).

There are immediate contradictions between the data analysis of Kousky and Wallace and the calculations described here. From (24) we see that there is a fixed relation between wave width and local vertical wavelength. In addition, there is a fixed relation between wave width and relative phase speed for a Kelvin wave [Eq. (58)]. These relations are displayed graphically in Fig. 8. In addition the relation between period and

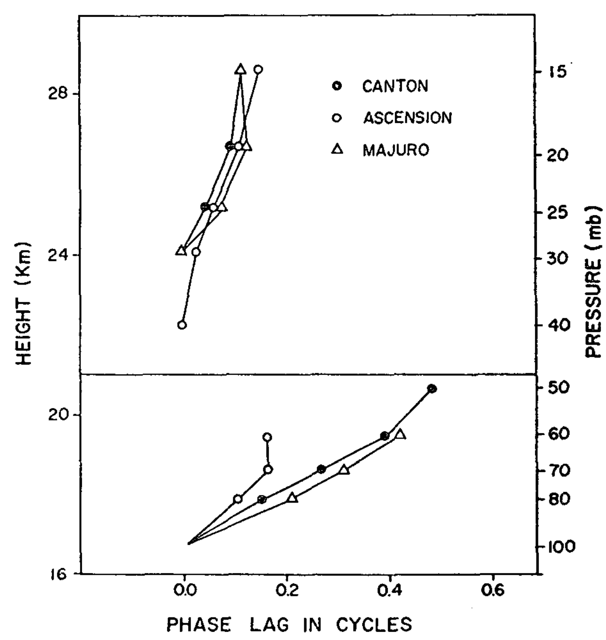

FIG. 7. Observed phase variation with height when Kelvin waves are observed (from Kousky and Wallace, 1971).

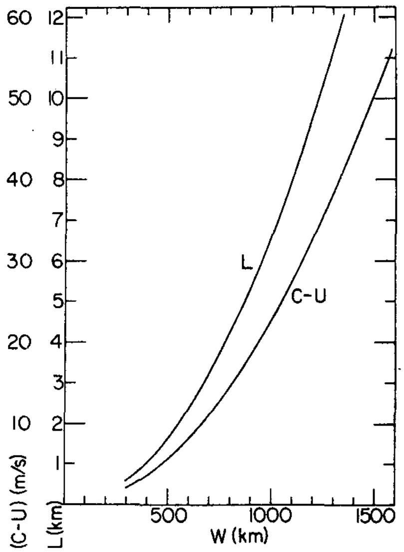

FIG. 8. Local vertical wavelength L and relative phase speed $c-U$ as a function of wave half-width W for equatorial Kelvin waves.

phase speed for zonal wavenumbers 1 and 2 is shown in Fig. 9. It is evident from Fig. 8 that decreasing width is associated with markedly decreasing vertical wavelength which is the opposite of what Kousky and Wallace found. Assuming errors in observed wave amplitude of ~ 1 m sec^{-1}, then the wave width[5] (as inferred from the observed latitudinal spread) at 26 km should be between about 444 and 750 km, and between 700 and 1300 km at 19 km. In what follows we shall assume that the disturbances described by Kousky and Wallace do indeed have widths in the ranges mentioned above. Then using Fig. 8 we see that $(c-U)$ must be between 11 and 38 m sec^{-1} at 19 km. Now, $U \approx -19$ m sec^{-1} at 19 km; hence, c must be between -8 and 19 m sec^{-1}. From Fig. 9 we see that this range for c is incompatible with s (the zonal wavenumber) $=1$, since periods outside the range 8–20 days have been filtered out of the data. Similarly $(c-U)$ must be between 4.5 and 13.0 m sec^{-1} at 26 km, and since $U \approx 12.5$ m sec^{-1} at 26 km, c must be between about 17 and 26 m sec^{-1}. This marginally allows $s=1$; however, the period would have to be between 18 and 20 days which is inconsistent with the observed power spectra.

In view of the above inconsistencies, I would like to try a different approach. I shall assume that both the upper and lower wave regimes are primarily a single wave. From the above discussion we see that c for such a wave must be between 17 and 19 m sec^{-1}. For $s=2$, this implies a period between 12 and 13.5 days in excellent agreement with Kousky and Wallace's power spectrum results. With the formulas of Section 4 and the distribution of U shown in Fig. 4 we may calculate the structure of such a wave. For α I have chosen 1/17.5 days on the basis of the calculations in Section 7a. I have, somewhat arbitrarily, chosen the amplitude of the zonal wind oscillation to be 7 m sec^{-1} at 16 km over the equator. In Fig. 10 we show the distribution with height of zonal wind amplitude at the equator, the wave width, the total vertical flux of zonal momentum, and the phase for such a Kelvin wave. Open circles indicate the levels at which radiosonde data were available. Several features are in acceptable agreement with the picture developed by Kousky and Wallace. For example, the calculations show a maximum in amplitude near 25 km, a vertical wavelength of about 8 km below 40 mb, and a rapid decrease of momentum flux in the westerly shear zone. However, the calculations predict rapidly decreasing vertical wavelengths in the region above 40 mb with vertical wavelengths between 2.5 and 1 km, not 35 km. It is important to note that the levels of observation above 40 mb are insufficiently close to resolve waves with such short wavelengths; this could plausibly account for the lack of correlation between phases measured above and below 40 mb. Also, if the waves in the atmosphere were really like the one in Fig. 10, and if observations were taken at the levels indicated, these observations would reveal little if any regular phase variation with height.

Another important discrepancy between Fig. 10 and Kousky and Wallace's description is that the latter has an amplitude maximum near 19 km which the present calculations do not suggest. Related to this is another discrepancy: my calculations show that, away from

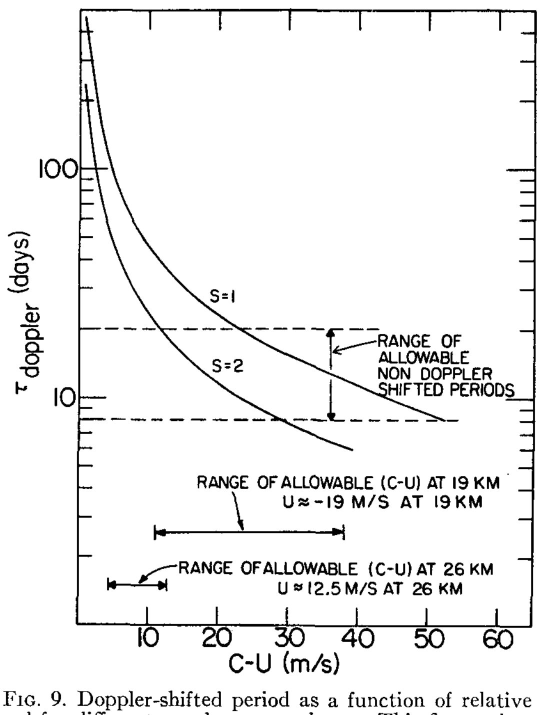

FIG. 9. Doppler-shifted period as a function of relative phase speed for different zonal wavenumbers s. This figure also relates c to the observed period. See text for details.

[5] The width corresponds to the e-folding distance for power density.

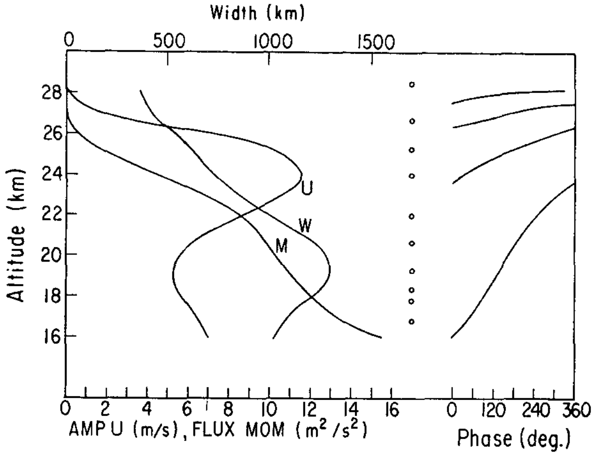

FIG. 10. The distribution with height of the amplitude of the zonal wind oscillation at the equator (U), the integrated vertical flux of zonal momentum (M), the latitudinal scale as measured from the equator (W), and the phase for a Kelvin wave propagating through a mean field given by U_2 in Fig. 4. Calculations are for $\alpha=1/17.5$ days. Also shown (by open circles) are the approximate levels from which Kousky and Wallace (1971) had radiosonde data.

sources of excitation the Kelvin wave (like any other wave whose phase velocity is westerly relative to the mean flow) carries only westerly momentum upward; however, Kousky and Wallace report a net deposition of easterly momentum by the waves below 19 km. This almost certainly indicates the presence of either forcing or instability in the region below 19 km, which, in turn, could explain the observed maximum in amplitude at 19 km.

The above discussion provides an alternative interpretation of the data used by Kousky and Wallace, an alternative, moreover, which is compatible with our theoretical ideas of equatorial waves.

8. Relevance to the quasi-biennial oscillation

In Lindzen and Holton's (1968) theory of the quasi-biennial oscillation, waves such as those described above were invoked as momentum sources for the quasi-biennial oscillation in the zonal wind in the tropical stratosphere. The only mechanism considered for momentum exchange between the waves and the mean flow was critical level absorption. As a result, they required the existence of a spectrum of waves with a continuous distribution of phase speeds between the maximum westerly and easterly velocities occurring in the quasi-biennial oscillation. A necessary consequence (for the theory, that is) of critical level absorption was that the levels of maximum easterly or westerly velocity in the quasi-biennial oscillation served as barriers to the further passage of equatorial planetary scale waves; this led to the downward progression of the levels of maximum velocity.

A consideration of the available data for the equatorial stratosphere shows rather unambiguously the existence of waves with phase speeds corresponding to the maximum easterly and westerly quasi-biennial velocities. However, there is virtually no evidence for the existence of all intermediate phase speeds. The results in this paper eliminate the need for such intermediate phase speeds; realistic levels of dissipation are sufficient to account for the deposition of mean zonal momentum throughout the region below a critical level. At the same time, levels of maximum easterly or westerly quasi-biennial velocity still act as barriers to the further passage of waves.

Acknowledgments. This work was supported by the National Science Foundation under Grant GA 25904.

APPENDIX

Evaluation of l_1

Our initial aim is to determine l_1 so that I_3 will be proportional to $\beta^2 l_0^2\xi^2-\omega^2$. I shall show how this is done for the Kelvin wave ($n=-1$). The procedure for the Yanai wave ($n=0$) is identical. From (29) we have

$$\Phi_0^{(-1)}=A_0^{(-1)}(\tau)e^{-\xi^2/2}e^{iz}, \tag{A1}$$

from which we can show that

$$\frac{\partial\Phi_0^{(-1)}}{\partial\xi}=-\xi\Phi_0^{(-1)}, \tag{A2}$$

$$\frac{\partial^2\Phi_0^{(-1)}}{\partial\xi^2}=(\xi^2-1)\Phi_0^{(-1)}. \tag{A3}$$

Using (A1)–(A3) it can be shown that

$$I_3=[A\xi^4+B\xi^2+C]iA_0(\tau)\exp(-\xi^2/2)\exp(i\hat{z}), \tag{A4}$$

where

$$A=2gS\beta^2\frac{\hat{l}_1}{l_0}, \tag{A5}$$

$$B=2gS\beta^2\hat{l}_1\left[\frac{k}{\hat{\omega}}l_0(\beta-k\hat{\omega})-\frac{1}{l_0}\right]-\beta U'f\beta^2 l_0, \tag{A6}$$

$$C=-\beta U'f\hat{\omega}^2, \tag{A7}$$

$$l_1=i\hat{l}_1. \tag{A8}$$

What we want is

$$A\xi^4+B\xi^2+C=(\beta^2l_0^2\xi^2-\hat{\omega}^2)(D\xi^2-E)=D\beta^2l_0^2\xi^4 +(-\hat{\omega}^2D-\beta^2l_0^2E)\xi^2+E\hat{\omega}^2. \tag{A9}$$

Collecting terms in (A9) we have

$$A=D\beta^2l_0^2, \tag{A10}$$

$$C=-\beta U'f\hat{\omega}^2, \tag{A11}$$

from which we get in turn

$$D=\frac{2\beta U'f}{\left[-1+\frac{k}{\hat{\omega}}l_0^2(\beta-k\hat{\omega})+\frac{\hat{\omega}^2}{\beta^2}\frac{1}{l_0^2}\right]}, \tag{A12}$$

$$E=-\beta U'f. \tag{A13}$$

From (A12), (A13) and

$$B=-\hat{\omega}^2D-\beta^2l_0^2E, \tag{A14}$$

we find

$$l_1=\frac{1}{gS}\frac{\beta l_0^3U'f}{\left[-1+\frac{k}{\hat{\omega}}l_0^2(\beta-k\hat{\omega})+\frac{\hat{\omega}^2}{\beta^2}\frac{1}{l_0^2}\right]}. \tag{A15}$$

Using (24) and (30), it can be shown that

$$-1+\frac{k}{\hat{\omega}}l_0^2(\beta-k\hat{\omega})+\frac{\hat{\omega}^2}{\beta^2}\frac{1}{l_0^2}=-2. \tag{A16}$$

Thus, (A13)–(A15) can be simplified to

$$D=E=-\beta U'\sqrt{\frac{S}{h}}, \tag{A17}$$

$$l_1=-i\frac{1}{2}\frac{U'}{\sqrt{gS}}l_0. \tag{A18}$$

Using (A9), (A4) and (A17) we have

$$\frac{I_3}{(\beta^2l_0^2\xi^2-\hat{\omega}^2)^2}=\frac{-i\beta U'\sqrt{\frac{S}{h}}(\xi^2-1)A_0\exp(-\xi^2/2)\exp(i\hat{z})}{(\beta^2l_0^2\xi^2-\hat{\omega}^2)}. \tag{A19}$$

Finally, using (A1), (A2), (A3) and (A18) in addition to (24) and (30), we can evaluate I_2, i.e.,

$$I_2=\left[\beta U'\sqrt{\frac{S}{h}}\xi^2-\beta U'\sqrt{\frac{S}{h}}\right]i\times\exp(-\xi^2/2)A_0(\tau)\exp(i\hat{z}), \tag{A20}$$
$$=-I_3/(\beta^2l_0^2\xi^2-\hat{\omega}^2).$$

Thus, $I_2/(\beta^2l_0^2\xi^2-\hat{\omega}^2)$ clearly cancels $I_3/(\beta^2l_0^2\xi^2-\hat{\omega}^2)^2$.

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