Equatorial Planetary Waves in Shear: Part II

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ABSTRACT

The general problem of the vertical propagation of equatorial waves through mean fields with vertical shear is solved analytically for all meridional wavenumbers using asymptotic multiple-scale methods. The results are used to show that the mixed gravity-Rossby wave is not the only easterly wave capable of penetrating the stratosphere, while the Kelvin wave is the only westerly wave capable of doing so. The results are also used to evaluate the effects of mean wind on diurnal tides.

1. Introduction

In Part I (Lindzen, 1971) multiple-scale analysis was used to obtain asymptotic expressions for the vertical propagation of internal equatorial Kelvin waves and mixed gravity-Rossby waves (referred to as Yanai waves) in a fluid whose basic state included a vertical shear. In the general context of equatorial wave theory the Kelvin and Yanai waves correspond to meridional wavenumbers $n=-1$ and $n=0$, respectively (Lindzen and Matsuno, 1968; Holton and Lindzen, 1968). The novelty of the analysis in Part I arose from the fact that the basic equations were nonseparable in their height and latitude dependence. As a result the Kelvin and Yanai waves changed horizontal scale as they propagated vertically. For wavenumbers $n\geqslant 1$, the possibility arose of waves changing both horizontal extent and form as they propagated vertically; this significantly complicated the analysis, and as a result, Part I did not include the analysis of waves for which $n\geqslant 1$. The extension of the earlier analysis to $n\geqslant 1$ will be described. The results will be used to show that easterly waves for which $n=1$ (there are two corresponding to internal gravity and Rossby waves) are about as effective as Yanai waves in propagating through the stratosphere. It will also be shown that the Kelvin wave is, indeed, the most efficient westerly wave. Finally, the wind dependence of the main propagating diurnal tidal mode (Lindzen, 1967a) will be evaluated.

2. Equations

As in Part I, the equations used are identical to those in Lindzen (1970). For the reader's convenience we shall repeat them here. We will consider linearized waves in a Boussinesq fluid on an equatorial β plane. Both perturbations and the basic state are taken to be in hydrostatic balance. The basic flow U is taken to be purely zonal, dependent only on height, and in geostrophic balance. The basic state is characterized by a constant stability S where

$$S\approx -\frac{\partial}{\partial z}(\ln\bar{\rho}), \tag{1}$$

where $\bar{\rho}$ is the basic density. The Richardson number of the basic state is assumed to be large. The dependence of the perturbations on time t and west-east distance x is taken to be of the form

$$e^{i(kx+\omega t)}, \tag{2}$$

where ω is the wave frequency, and k the zonal wavenumber. The resulting perturbation equations are

$$(i\omega+ikU+\alpha)u+w\frac{dU}{dz}=-ik\Phi+\beta yv, \tag{3}$$

$$(i\omega+ikU+\alpha)v=-\frac{\partial\Phi}{\partial y}-\beta yu, \tag{4}$$

$$\frac{\partial\Phi}{\partial z}=-g\frac{\delta\rho}{\bar{\rho}}, \tag{5}$$

$$(i\omega+ikU+\alpha)\frac{\delta\rho}{\bar{\rho}}-wS+\frac{\beta y}{g}\frac{dU}{dz}v=0, \tag{6}$$

$$iku+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{7}$$

where

$$\Phi\equiv\frac{\delta p}{\bar{\rho}}; \tag{8}$$

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α is an arbitrary damping coefficient; u, v, w, δp and $\delta\rho$ are the perturbation zonal, southerly, and vertical velocities, and pressure and density, respectively; y and z the southerly and vertical distances; β is $2\Omega/a$; a and Ω the earth's radius and rotation rate; and overbars refer to the basic state. Taking the Richardson number as

$$\mathrm{Ri}=gS\Big/\left(\frac{dU}{dz}\right)^2, \tag{9}$$

and dropping terms $O(\mathrm{Ri}^{-1})$ compared to 1, we can derive from Eqs. (3)–(7) a single equation for Φ:

$$\begin{aligned}(\beta^2y^2-\hat{\omega}^2)^2\frac{\partial^2\Phi}{\partial z^2}&+2\beta y(\beta^2y^2-\hat{\omega}^2)\frac{dU}{dz}\frac{\partial^2\Phi}{\partial z\partial y}\\ &+gS(\beta^2y^2-\hat{\omega}^2)\frac{\partial^2\Phi}{\partial y^2}+\beta\frac{dU}{dz}\left[\frac{\beta y^2}{gS}(\beta^2y^2-\hat{\omega}^2)\frac{d^2U}{dz^2}\right.\\ &\left.-(\beta^2y^2+\hat{\omega}^2)\right]\frac{\partial\Phi}{\partial z}-2gS\beta^2y\frac{\partial\Phi}{\partial y}\\ &+\frac{gSk}{\hat{\omega}}[\beta(\beta^2y^2+\hat{\omega}^2)-k\hat{\omega}(\beta^2y^2-\hat{\omega}^2)]\Phi=0,\end{aligned} \tag{10}$$

where

$$\hat{\omega}=\omega+kU-i\alpha.$$

We shall adopt the convention that $\omega>0$, in which case westerly waves are associated with negative k, and easterly waves with positive k.

Details may be found in Lindzen (1970). Other fields are related to Φ by the following equations:

$$v=(\beta^2y^2-\hat{\omega}^2)^{-1}\left[ik\beta y\Phi-i\hat{\omega}\left(\frac{\beta y}{gS}\frac{dU}{dz}\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\right)\Phi\right], \tag{11}$$

$$\frac{\delta\rho}{\bar{\rho}}=-\frac{1}{g}\frac{\partial\Phi}{\partial z}, \tag{12}$$

$$w=\frac{1}{gS}\left(-i\hat{\omega}\frac{\partial\Phi}{\partial z}+\beta y\frac{dU}{dz}v\right), \tag{13}$$

$$u=-\frac{1}{\beta y}\left(\frac{\partial\Phi}{\partial y}+i\hat{\omega}v\right). \tag{14}$$

The Boussinesq approximation is by no means necessary, though it helps simplify an already cumbersome problem. Its main effect is to eliminate a factor $\bar{p}^{-\frac{1}{2}}$ that would otherwise appear in the solutions for u, v, w, $\delta\rho/\bar{\rho}$ and Φ. The results are otherwise similar provided that S is identified with $(1/\bar{T})[\partial\bar{T}/\partial z)+(g/c_p)]$ in the atmosphere.

Our boundary conditions on Φ will be $\Phi\to 0$ as $y^2\to\infty$. We will assume that a wave is forced at $z=0$, and only the upward travelling wave will be considered.

3. Mathematical solution

Again, as in Part I, I wish to obtain asymptotic solutions to (10) by means of the "two-variable" technique (otherwise known as the multiple-scale technique) described in Cole (1968). Once more it proves convenient to repeat portions of the analysis developed in Part I. The technique requires that the characteristic scale for the variation of $U(z)$ must be much larger than the characteristic vertical scale of the waves we are solving for (i.e., the local vertical wavelength divided by 2π). The next step is to replace z by "slow" and "fast" height variables (U being taken to be a function solely of the slow variable), where the slow variable is of the form

$$\tau=\epsilon z,\quad \epsilon\ll 1, \tag{15}$$

where ϵ is chosen so that the τ derivatives of U are of order unity. Then

$$\frac{dU}{dz}=\epsilon\frac{dU}{d\tau}=\epsilon U', \tag{16a}$$

$$\frac{d^2U}{dz^2}=\epsilon^2\frac{d^2U}{d\tau^2}=\epsilon^2U''. \tag{16b}$$

For our fast variable we choose

$$\hat{z}=\int_0^z f(\tau)dz, \tag{17}$$

where f may be expanded as

$$f=f_0(\tau)+\epsilon^2f_2(\tau)+\epsilon^3f_3(\tau)+\cdots. \tag{18}$$

The order ϵ term in (18) is omitted; its effects are already included in the τ dependence; and $f(\tau)$ is determined in the course of solution. In addition, because (10) is nonseparable in its z and y dependence, it proves convenient to replace y by a scaled variable

$$\xi=y/l(\tau), \tag{19}$$

where

$$l=l_0(\tau)+\epsilon l_1(\tau)+\cdots, \tag{20}$$

and where $l(\tau)$ is also determined in the course of solution. In terms of our new variables

$$\Phi=\Phi(\hat{z},\xi,\tau), \tag{21}$$

where Φ is also expanded in powers of ϵ:

$$\Phi=\Phi_0(\hat{z},\xi,\tau)+\epsilon\Phi_1(\hat{z},\xi,\tau)+\cdots. \tag{22}$$

The change of variables in (10) and the substitution of expansions (18), (20) and (22) leads to a very lengthy equation which is described in detail in Part I. The resulting equation may be ordered according to powers of ϵ. To zeroth order one gets

$$(\beta^2\xi^2 l_0^2-\hat{\omega}^2)f_0^2\frac{\partial^2\Phi_0}{\partial \hat{z}^2}+\frac{gS}{l_0^2}(\beta^2\xi^2l_0^2-\hat{\omega}^2)\frac{\partial}{\partial\xi}\left[(\beta^2\xi^2l_0^2-\hat{\omega}^2)^{-1}\frac{\partial\Phi_0}{\partial\xi}\right]+\frac{gSk}{\hat{\omega}}[\beta(\beta^2\xi^2l_0^2+\hat{\omega}^2)-k\hat{\omega}(\beta^2\xi^2l_0^2-\hat{\omega}^2)]\Phi_0=0. \quad (23)$$

Eq. (23) involves τ only parametrically, and its solutions are simply those given in Lindzen (1967a), Lindzen and Matsuno (1968) and Holton and Lindzen (1968). There are a countably infinite set of these solutions designated by $n=-1,0,1,2,\ldots$. For each of the solutions, the scaling factors f_0 and l_0 turn out to be related simply as

$$\beta^2 l_0^4=\frac{gS}{f_0^2}=gh^{(n)}, \quad (24)$$

where $h^{(n)}$, the equivalent depth of tidal theory (*viz.* Chapman and Lindzen, 1970), is determined for each n. *This means that $\hat{z}$ and ξ represent different scalings for each n.* Once $\Phi_0^{(n)}$ is obtained, the remaining fields to zeroth order in ϵ are given by [see Eqs. (11)–(14)]

$$v_0=(\beta^2l_0^2\xi^2-\hat{\omega}^2)^{-1}\left(ik\beta l_0\xi\Phi_0-i\hat{\omega}\frac{1}{l_0}\frac{\partial\Phi_0}{\partial\xi}\right), \quad (25)$$

$$w_0=\frac{1}{gS}\hat{\omega}f\Phi_0, \quad (26)$$

$$u_0=-\frac{1}{\beta l_0\xi}\left(\frac{1}{l_0}\frac{\partial\Phi_0}{\partial\xi}+i\hat{\omega}v_0\right), \quad (27)$$

$$\frac{\delta\rho_0}{\bar{\rho}}=-\frac{i}{g}f\Phi_0. \quad (28)$$

The solutions for $n=-1$ (Kelvin waves) and $n=0$ (Yanai waves) have been discussed in detail in Part I. I shall, here, restrict myself to $n\geqslant 1$. The zeroth order solutions for $n\geqslant 1$ are discussed in Lindzen (1967a) and Lindzen and Matsuno (1968), and given by

$$\Phi_0^{(n)}=A_0^{(n)}(\tau)[-2nq(\tau)H_{n-1}(\xi)+H_{n+1}(\xi)]\times\exp(-\xi^2/2)\exp(i\hat{z}), \quad (29)$$

$$\sqrt{gh^{(n)}}=\frac{a\Omega(2n+1)}{s^2\left[\left(\frac{2\Omega}{s\hat{\omega}}\right)-1\right]}\times\left\{1\pm\left[1-\left(\frac{\hat{\omega}}{\Omega}\right)^2\left(\frac{s}{2n+1}\right)^2\left(\frac{2\Omega}{s\hat{\omega}}-1\right)\right]^{\frac{1}{2}}\right\}, \quad (30)$$

$$q(\tau)=\frac{1+\left(\frac{k}{\hat{\omega}}\right)\sqrt{gh^{(n)}}}{1-\left(\frac{k}{\hat{\omega}}\right)\sqrt{gh^{(n)}}}, \quad (31)$$

where $H_n(\xi)$ is the nth Hermite polynomial in ξ. In addition to the solutions given by (20), there are also solutions proportional to $e^{-i\hat{z}}$; however, these solutions represent downward propagating wave energy. The minus sign in (30) corresponds to internal gravity wave-type solutions; the plus sign corresponds to Rossby-type waves. Vertically propagating solutions of the latter type exist only when

$$\frac{2\Omega}{s\hat{\omega}}>1 \quad (30a)$$

(*viz.* Lindzen and Matsuno, 1968). However, even when (30a) is satisfied, Rossby waves are frequently (especially for large n) associated with horizontal scales larger than the pole-equator distance. This is, of course, a fictitious result arising from the inadequacy of the unbounded equatorial beta-plane to describe such modes.

For $n\geqslant 1$, since $q(\tau)\neq 0$, our solutions are no longer simply factorizable into functions of τ, z and ξ alone. This factorizability is essential to the usual application of "two-variable" techniques. Therefore, for $n\geqslant 1$ a somewhat new approach is needed wherein q is expanded in powers of ϵ, i.e.,

$$q(\tau)=q_0(\tau)+\epsilon q_1(\tau)+\cdots, \quad (32)$$

where

$$q_0(\tau)=\frac{\left(\frac{k}{\hat{\omega}}\right)\sqrt{gh^{(n)}}}{1-\left(\frac{k}{\hat{\omega}}\right)\sqrt{gh^{(n)}}}. \quad (33)$$

Using (32) it becomes convenient to rewrite (22) as

$$\Phi^{(n)}=\Phi_0^{(n)}+\epsilon(\psi_1^{(n)}+\hat{\Phi}_1^{(n)})+\cdots, \quad (34)$$

where

$$\psi_m^{(n)}=-2nA_0^{(n)}(\tau)q_m(\tau)H_{n-1}(\xi)e^{-\xi^2/2}e^{i\hat{z}}. \quad (35)$$

Then, to first order in ϵ, we have

$$\frac{f_0^2l_0^2}{gS}\frac{\partial^2\hat{\Phi}_1^{(n)}}{\partial\hat{z}^2}+\frac{\partial}{\partial\xi}\left\{(\beta^2\xi^2l_0^2-\hat{\omega}^2)^{-1}\frac{\partial\Phi_1^{(n)}}{\partial\xi}\right\}+\frac{l_0^2k}{\hat{\omega}}\left\{\beta\frac{(\beta^2\xi^2l_0^2+\hat{\omega}^2)}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)}-k\hat{\omega}(\beta^2\xi^2l_0^2-\hat{\omega}^2)^{-1}\right\}\Phi_1^{(n)}=-\frac{l_0^2}{gS}\left\{I_1+J_1+\frac{I_2+J_2}{(\beta^2l_0^2\xi^2-\hat{\omega}^2)}+\frac{I_3+J_3}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)^2}\right\}, \quad (36)$$

where

$$I_1=2f_0\frac{\partial^2\Phi_0}{\partial\hat{z}\partial\tau}+\frac{df_0}{d\tau}\frac{\partial\Phi_0}{\partial\hat{z}}-2\xi\frac{f_0}{l_0}\frac{dl_0}{d\tau}\frac{\partial^2\Phi_0}{\partial\hat{z}\partial\xi},\quad(37)$$

$$I_2=4l_0l_1\beta^2\xi^2f_0^2\frac{\partial^2\Phi_0}{\partial\hat{z}^2}+2\beta\xi\bar{u}'f_0\frac{\partial^2\Phi_0}{\partial\hat{z}\partial\xi}-2gS\frac{l_1}{l_0^3}\frac{\partial^2\Phi_0}{\partial\xi^2},\quad(38)$$

$$I_3=2gS\beta^2\xi^2\frac{l_1}{l_0}\frac{\partial^2\Phi_0}{\partial\xi^2}-\beta\bar{u}'f_0(\beta^2\xi^2l_0^2+\hat{\omega}^2)\frac{\partial\Phi_0}{\partial\hat{z}}+\frac{gSk}{\hat{\omega}}2l_0l_1\beta^2\xi^2(\beta-k\hat{\omega})\Phi_0,\quad(39)$$

$$J_1=f_0^2\frac{\partial^2\psi_1}{\partial\hat{z}^2},\quad(40)$$

$$J_2=\frac{gS}{l_0^2}\frac{\partial^2\psi_1}{\partial\xi^2}-gSk^2\psi_1,\quad(41)$$

$$J_3=-\frac{gS}{l_0^2}2\beta^2l_0^2\xi\frac{\partial\psi_1}{\partial\xi}+gS\frac{k}{\hat{\omega}}\beta(\beta^2l_0^2\xi^2+\hat{\omega}^2)\psi_1.\quad(42)$$

Just as in Part I, we must use our available parameters, l_1, q_1 and A_0 in order to (i) eliminate the singularities on the right-hand side of (36), and (ii) suppress any projection of the right-hand side of (36) on Φ_0. Because of the more complex nature of $\Phi_0^{(n)}$ here, however, the task is more complicated than it was in Part I. The results, however, are comparably simple.

To begin with, I substitute expressions (29) and (35) for $\Phi_0^{(n)}$ and $\psi_1^{(n)}$ into (38), (39), (41) and (42) to get

$$(I_2+J_2)(\beta^2\xi^2l_0^2-\hat{\omega}^2)+I_3+J_3$$
$$=iA_0\beta^2l_0^2q_0e^{iz}e^{-\xi^2/2}(1-K^2)^{-2}\{(\sum_{m=0}^{2}a_m\xi^{2m})H_{n-1}(\xi)+(\sum_{m=0}^{2}b_m\xi^{2m})H_{n+1}(\xi)\},\quad(43)$$

where

$$K=\frac{k}{\hat{\omega}}\beta l_0^2=\frac{k}{\hat{\omega}}\sqrt{gh},\quad(44)$$

$$a_0=-2n\{2M[2m(K^4-2K^3+2n+1)+(1-K)(-2K^3+2K^2+K-1)]+(2n+1-K)[-2(2n-1)(1-K^2)L+(2(n+1)(1-K)^2+(2n-1)(1-K^2))Q]\},\quad(45)$$

$$a_1=2n\{4(1-K^2)(2n+1-K)M-2L[(2n+1)(3-K^2)(1-K^2)-2K(1-K^2)]+Q[2(n+1)(1-K)^2(1-K^2)+((2n+1)(3-K^2)(1-K^2)-2K(1-K^2))]\},\quad(46)$$

$$a_2=4n(1-K^2)^2[-M+2L-Q],\quad(47)$$

$$b_0=-\{4n(1-K^2)^2M-(2n+1-K)[2(2n+3)(1-K)^2L+(2n(1-K^2)+(2n+3)(1-K)^2)Q]\},\quad(48)$$

$$b_1=\{2L(1-K)^2[(2n+1)(3-K^2)-2K]+Q[(1-K)^2\times((2n+1)(3-K^2)-2K)+2n(1-K^2)^2]\},\quad(49)$$

$$b_2=-2(1-K^2)(1-K)^2\{2L+Q\},\quad(50)$$

and

$$M=\tfrac{1}{2}\beta^2f_0^2l_0^2\frac{\hat{q}_1}{q_0},\quad(51)$$

$$L=\beta^2f_0^2\hat{l}_1,\quad(52)$$

$$Q=\beta f_0u_0',\quad(53)$$

$$\hat{l}_1=-il_1,\quad(54)$$

$$\hat{q}_1=-iq_1.\quad(55)$$

In arriving at the above, the formal dependence of the coefficients on $\hat{\omega}$ (apart from the dependence on K, which depends on both k and $\hat{\omega}$) has been eliminated by use of the dispersion relation which can be conveniently written

$$\Omega^2(1-K^2)=2n+1-K,\quad(56)$$

where

$$\Omega=\frac{\hat{\omega}}{\beta l_0}.\quad(57)$$

In terms of Ω and K, q_0 has the simple form

$$q_0=\frac{1+K}{1-K}.\quad(58)$$

The right-hand side (r.h.s.) of (43) is a polynomial of order $n+5$ consisting in $(n+5)/2$ terms (if n is odd) or $(n+6)/2$ terms (if n is even).

I now impose the condition that the r.h.s. of (36) is *non-singular*. This, in turn, requires that the r.h.s. of (43) be proportional to $(\beta^2l_0^2\xi^2-\hat{\omega}^2)^2$, or, equivalently, to $(\xi^2-\Omega^2)^2$. Once more, Ω^2 may be eliminated by use of (56). More explicitly, let us call the bracketed portion of the r.h.s. of (43), $\mathfrak{L}$; then we want

$$\mathfrak{L}=(\xi^2-\Omega^2)^2(1-K^2)^2\sum^{n+1}\mu_mH_m(\xi)$$
$$=[\xi^2(1-K^2)-(2n+1-K)]^2\sum^{n+1}\mu_mH_m(\xi).\quad(59)$$

Both $\mathfrak{L}$ and the r.h.s. of (59) are of the same order in ξ. However, we have only $(n+1)/2$ μ_m's to choose in order to make the coefficients of $(n+5)/2$ terms equal. Fortunately, we also have M and L (or equivalently q_1

and l_1) at our disposal, and the system is, therefore, determinate; i.e., it is, in general, possible to suppress the singular part of the l.h.s. of (36).

It is necessary for what follows to write down the first few equations for the μ_m's. We first require expressions for the Hermite polynomials

$$H_n(\xi)=\frac{n!2^n\xi^n}{n!}-\frac{n!2^{n-2}}{(n-2)!}\xi^{n-2}+O(\xi^{n-4}). \qquad (60)$$

Thus,

$$\begin{aligned}\mathfrak{L}=b_2 2^{n+1}\xi^{n+5}&\\ +[-(n+1)n2^{n-1}b_2+2^{n+1}b_1+2^{n-1}a_2]\xi^{n+3}&\\ +O(\xi^{n+1}),&\qquad (61)\end{aligned}$$

and the r.h.s. of (43) is given by

$$\begin{aligned}\text{r.h.s. of (43)}&=2^{n+1}(1-K^2)^2\mu_{n+1}\xi^{n+3}\\ &+2^{n-1}\left\{(1-K^2)^2\mu_{n-1}+\left[-(1-K^2)^2\frac{(n+1)!}{(n-1)!}\right.\right.\\ &\left.\left.-8(2n+1-K)(1-K^2)\right]\mu_{n+1}\right\}\xi^{n+3}+O(\xi^{n+1}). \qquad (62)\end{aligned}$$

Thus, the first two equations which $\{\mu_m\}$ and L, M must satisfy in order that singular forcing at $O(\epsilon)$ be suppressed are

$$b_2=(1-K^2)^2\mu_{n+1}, \qquad (63)$$

and

$$\begin{aligned}-(n+1)nb_2+4b_1+a_2&\\ =(1-K^2)^2\mu_{n-1}+\left[-(1-K^2)^2\frac{(n+1)!}{(n-1)!}\right.&\\ \left.-8(2n+1-K)(1-K^2)\right]\mu_{n+1},&\qquad (64)\end{aligned}$$

or using Eqs. (50), (49) and (47),

$$-2(1-K)^2(2L+Q)=(1-K^2)\mu_{n+1}, \qquad (65)$$

and

$$\begin{aligned}\{n(n+1)(1-K^2)+2[(2n+1)(3-K^2)-2K]&\\ +2n(1+K^2)\}2(1-K)^2(2L+Q)-4n(1-K^2)^2M&\\ =(1-K^2)^2\mu_{n-1}+\left[-(1-K^2)^2\frac{(n+1)!}{(n-1)!}\right.&\\ \left.-8(2n+1-K)(1-K^2)\right]\mu_{n+1}.&\qquad (66)\end{aligned}$$

Of course, to determine $\{\mu_m\}$ and L and M we need the remaining equations as well. However, our first task is to determine A_0 in order to suppress any part of the r.h.s. of (36) which has finite projection of Φ_0 and which would, therefore, lead to spurious secular growth with respect to ξ. We may note in this regard that only those terms in (59) proportional to $H_{n+1}(\xi)$ and $H_{n-1}(\xi)$ will have projections of $\Phi_0^{(n)}$. The remaining terms will have no projection due to the orthogonality properties of Hermite polynomials. The projection of any function on $\Phi_0^{(n)}$ is given by

$$\text{Proj. } g \text{ on } \Phi_0^{(n)}=\int_{-\infty}^{\alpha}\Phi_0^{(n)}g\,d\xi\Big/\int_{-\infty}^{\infty}\Phi_0^{(n)2}d\xi, \qquad (67)$$

where

$$\int_{-\infty}^{\infty}e^{-\xi^2}H_n(\xi)H_m(\xi)d\xi=\delta_{mn}2^n n!\sqrt{\pi}, \qquad (68)$$

and therefore

$$\int_{-\infty}^{\infty}\Phi_0^{(n)2}d\xi=2A_0^2e^{2i\hat{z}}2^n n!\sqrt{\pi}\{nq_0^2+(n+1)\}. \qquad (69)$$

Following the procedure of Part I we find

$$\begin{aligned}\text{Proj. of } I_1=2if_0\left\{\frac{1}{A_0}\frac{dA_0}{d\tau}+n\frac{dq_0}{d\tau}\frac{q_0}{\{nq_0^2+(n+1)\}}\right.&\\ \left.+\frac{1}{2}\frac{1}{f_0}\frac{df_0}{d\tau}+\frac{1}{2}\frac{1}{l_0}\frac{dl_0}{d\tau}\right\},&\qquad (70)\end{aligned}$$

which may be rewritten

$$\text{Proj. of } I_1=2if_0\frac{d}{d\tau}\left\{\ln\left[A_0\left[\left(q_0^2+\frac{n+1}{n}\right)f_0l_0\right]^{\frac{1}{2}}\right]\right\}. \qquad (71)$$

Eq. (71) differs from the corresponding expression in Part I due to the presence of the factor $[q_0^2+(n+1)/n]^{\frac{1}{2}}$. Similarly, the projection of J_1 is given by

$$\begin{aligned}\text{Proj. } J_1&=-\frac{nq_0q_1f_0^2}{\{nq_0^2+(n+1)\}}\\ &=-i\frac{nq_0f_0^2}{\{nq_0^2+(n+1)\}}\hat{q}_1. \qquad (72)\end{aligned}$$

To find the remaining projection we note that

$$\begin{aligned}P&\equiv\frac{(I_2+J_2)(\beta^2l_0^2\xi^2-\hat{\omega}^2)+(I_3+J_3)}{(\beta^2\xi^2l_0^2-\hat{\omega}^2)^2}\\ &=\frac{iA_0e^ie^{\hat{z}-\xi^2/2}}{\beta^2l_0^2}q_0\{\mu_{n+1}H_{n+1}+\mu_{n-1}H_{n-1}+\cdots\}, \qquad (73)\end{aligned}$$

and the projection of P on Φ_0 is given by

$$\text{Proj. } P=\frac{iq_0}{\beta^2l_0^2}\frac{\{2(n+1)\mu_{n+1}-q_0\mu_{n-1}\}}{2\{nq_0^2+(n+1)\}}. \qquad (74)$$

Now, we shall use (65) and (66) to eliminate $(2L+Q)$ and obtain the following relation between M and μ_{n+1} and μ_{n-1}:

$$-4nq_0M=-2n\beta^2f_0{}^2l_0{}^2\hat{q}_1 =q_0\mu_{n-1}-2(n+1)\mu_{n+1}. \quad (75)$$

Thus,

$$\text{Proj. } P=iq_0\frac{nf_0{}^2\hat{q}_1}{\{nq_0{}^2+(n+1)\}}=-\text{Proj. } J_1. \quad (76)$$

This leads us to an important generalization of the finding in Part I that for $n=-1,0$ all contributions to the r.h.s. of (36) *except* I_1 went to zero when the singularity was suppressed: namely, for $n\geqslant 1$ the *projection* of everything *except* I_1 on $\Phi_0{}^{(n)}$ goes to zero when the singularities are suppressed. This time, however, the remaining terms, themselves, do not necessarily go to zero; they may force $O(\epsilon)$ terms. Still we can now determine A_0 *without* explicitly solving for $\{\mu_m\}$, L, M. We do this by requiring

$$\text{Proj. } I_1=0,$$

which implies

$$\frac{d}{dr}\ln\left\{A_0\left[\left(q_0{}^2+\frac{n+1}{n}\right)f_0l_0\right]^{\frac{1}{2}}\right\},$$

or (77)

$$A_0=\frac{\text{const.}}{\left[\left(q_0{}^2+\frac{n+1}{n}\right)f_0l_0\right]^{\frac{1}{2}}}.$$

It should be noted that our solution for A_0 is independent of those terms in (10) proportional to $d\bar{u}/dz$.

4. Review of zeroth-order solutions

In this section I list the results obtained for various fields to zeroth order in ϵ:

$$\Phi_0{}^{(n)}=\frac{\text{const.}}{\left[\left(q_0{}^2+\frac{n+1}{n}\right)f_0l_0\right]^{\frac{1}{2}}} \times\{-2nq_0H_{n-1}(\xi)+H_{n+1}(\xi)\}e^{-\xi^2/2}e^{i\hat{z}}, \quad (78)$$

where

$$\xi\approx\frac{y}{l_0},$$

$$\hat{z}\approx\int_0^z f_0dz,$$

and f_0, l_0, q_0 (as well as $\sqrt{gh}$) have already been specified.

For most practical applications the $O(\epsilon)$ corrections are very small, with relative magnitudes $O(\text{Ri}^{-\frac{1}{2}})$; in the equatorial stratosphere $\text{Ri}\sim O(100)$.

To zeroth order, our solutions for other fields are

$$v_0\approx(\beta^2l_0{}^2\xi^2-\hat{\omega}^2)^{-1}\left(ik\beta l_0\xi\Phi_0-i\hat{\omega}\frac{1}{l_0}\frac{\partial\Phi_0}{\partial\xi}\right), \quad (79)$$

$$u_0\approx-\frac{1}{\beta l_0\xi}\left(\frac{1}{l_0}\frac{\partial\Phi_0}{\partial\xi}+i\hat{\omega}v_0\right), \quad (80)$$

$$w_0\approx\frac{\hat{\omega}}{gS}f_0\Phi_0, \quad (81)$$

$$\frac{\delta\rho_0}{\bar{\rho}}=-\frac{i}{g}f_0\Phi_0. \quad (82)$$

If we substitute (78) into (79) and (80) we get

$$v_0{}^{(n)}=\frac{2i\hat{\omega}\left(1+\frac{k}{\hat{\omega}}\beta l_0{}^2\right)}{\beta^2l_0{}^3}A_0H_n(\xi)e^{-\xi^2/2}e^{i\hat{z}}, \quad (83)$$

$$u_0{}^{(n)}=\frac{1}{\beta l_0{}^2}A_0\{2nq_0H_{n-1}(\xi)+H_{n+1}(\xi)\}e^{-\xi^2/2}e^{i\hat{z}}, \quad (84)$$

where

$$A_0=\frac{\text{const.}}{\left[\left(q_0{}^2+\frac{n+1}{n}\right)f_0l_0\right]^{\frac{1}{2}}}. \quad (84a)$$

5. Vertical fluxes

My results for vertical fluxes of momentum and for the wave action are similar to those obtained in Part I. For the vertical flux of zonal momentum due to linearized waves in a rotating fluid I again use the expression

$$F_m=\bar{\rho}(\overline{uw}-f\overline{\eta w}), \quad (85)$$

where overbar refers to a time (or longitude) average,

$$\eta=\frac{iv}{\hat{\omega}}$$

is the southerly displacement associated with the wave, and f the local Coriolis parameter. On an equatorial beta-plane

$$f=\beta y\approx\beta\xi/l_0.$$

As before, I find no useful theorems for the behavior of F_m; however, the behavior of $\langle F_m\rangle$, where $\langle\ \rangle$ refers to the integral from $y=-\infty$ to $y=+\infty$, is analogous to the behavior of F_m in a plane rotating fluid. Since we are dealing with a Boussinesq fluid we will consider $\tilde{F}_m=F_m/\bar{\rho}$ rather than F_m. The following results are readily obtained:

$$\overline{\langle uw\rangle}=\frac{\hat{\omega}}{gS}A_0{}^2\frac{f_0}{\beta l_0}2^n nn!\sqrt{\pi}\left\{-q_0{}^2+\frac{n+1}{n}\right\},\tag{86}$$

$$\overline{\langle f\eta w\rangle}=\frac{\left(1-\frac{k}{\hat{\omega}}\beta l_0{}^2\right)}{\beta l_0}A_0{}^2\frac{\hat{\omega}}{gS}f_0 2^n nn!\sqrt{\pi}\times\left\{-q_0{}^2+\frac{n+1}{n}q_0\right\},\tag{87}$$

$$\langle\tilde{F}_m\rangle=-k\frac{\text{const.}^2}{gS}2^n nn!\sqrt{\pi}\neq f_{cn}(z).\tag{88}$$

It may also be shown that the integrated wave action

$$\langle A\rangle=\frac{\overline{\langle w\Phi\rangle}}{\hat{\omega}}=-\langle\tilde{F}_m\rangle\frac{1}{k}.\tag{89}$$

The independence of $\langle\tilde{F}_m\rangle$ and $\langle A\rangle$ of z holds only when f_0 is real or equivalently in the absence of damping.[2]

There is some reason to suspect that the constancy with height of $\langle A\rangle$ and $\langle F_m\rangle$ in the absence of damping are fundamental properties of internal waves. If this is the case one may conjecture that in a fluid where S varied slowly with height, the constant in (78) and subsequent equations should be replaced by a constant times $(gS)^{\frac{1}{4}}$. Similarly, in a fluid where $\bar{\rho}$ varied significantly with z (but on a scale comparable with our "slow" variable) we would expect a factor $\bar{\rho}^{-\frac{1}{2}}$ in (78) and (84a). In the preceding flux relations we would then consider $\langle F_m\rangle$ rather than $\langle\tilde{F}_m\rangle$ and $\langle\overline{w\delta p}\rangle$ rather than $\langle\overline{w\Phi}\rangle$. The detailed investigation of the above conjecture still remains to be done.

6. Introduction of damping

The results in Section 5 assumed $\alpha=0$ in (3), (4) and (6). If α is sufficiently small the main effect of damping is to make f_0 complex leading to the exponential decay of our fields as

$$\exp\left(-\int_0^z g_0 dz\right),\tag{90}$$

where $g_0=$imag. part of f_0. The possibility was discussed in Part I that the damping might be entirely due to infrared cooling, i.e., $\alpha\neq 0$ in (6), but $\alpha=0$ in (3) and (4). With $\alpha\neq 0$ only in (6), the analysis proceeds identically to the problem without damping, except that in evaluating the vertical structure $f_0{}^2$ is replaced by

$$f_0{}^2\frac{(\omega+ku_0)}{(\omega+ku_0-i\alpha)}\tag{91}$$

(*viz.* Lindzen and McKenzie, 1967) in which case, for small α, f_0 is replaced by

$$f_0\left(1+\frac{1}{2}\frac{i\alpha}{\omega+ku_0}\right),\tag{92}$$

and

$$g_0\approx\frac{1}{2}\frac{\alpha}{\omega+ku_0}f_0,\tag{93}$$

where f_0 is calculated for a fluid *without* damping. The analysis leading to the above result is essentially given by Lindzen and McKenzie (1967).

When damping is introduced, the various vertical fluxes described in Section 5 decay with height as

$$\exp\left(-2\int_0^z g_0 dz\right).\tag{94}$$

The fact that damping proves important for internal equatorial waves makes the choice of α comparably important. In Part I, the question was asked whether there existed a choice of α (where equal frictional and thermal damping was considered) which would produce the best correspondence between calculations and observations of the vertical variations in amplitude of the mixed gravity-Rossby (Yanai) mode ($n=0$). It was concluded that $\alpha=1/17.5$ days led to such correspondence. However, the effects of damping depend critically on the Doppler-shifted phase speed of the wave, and in Part I where an approximate analytic representation was used for the mean wind, significant inaccuracies entered this quantity. For example the observed value of $\bar{u}$ at 25 km (*viz.* Maruyama, 1967) was -18.5 m sec^{-1} while $\bar{u}=-14$ m sec^{-1} was used in Part I. Equivalently $\bar{u}-c=4.5$ m sec^{-1} was observed while $\bar{u}-c=7$ m sec^{-1} was used in Part I. When the observed distribution of $\bar{u}$ is employed, the best correspondence between theory and observation is obtained for $\alpha=1/42$ days. Forty-two days is considerably longer than current estimates for the radiative relaxation time at 25 km. However, as has been pointed out earlier in this section, when we have damping due exclusively to Newtonian cooling the effective damping is less for any given choice of α than when we have both cooling and friction. Seeking a best correspondence with only thermal damping leads to a choice for α of 1/10 days which is in the range of existing estimates for the Newtonian cooling rate (Dickinson, 1968). As a final remark, I feel it essential to point out that the sensitivity of the choice of α to the distribution of $\bar{u}-c$ coupled with observa-

[2] When f_0 is real, if two fields are given by $A=ae^{i(z+kx+\omega t)}$ and $B=be^{i(z+kx+\omega t+\varphi_{ab})}$ (where φ_{ab} is a phase difference between the two fields), then $\overline{AB}=(ab/2)\cos\phi_{ab}$. If, however, f_0 is complex then $\hat{z}=\hat{z}_r+i\hat{z}_i$ and $\overline{AB}=(ab/2)e^{-2\hat{z}_i}\cos\phi_{ab}$. The realness of f_0 has been assumed for (88) and (89).

tional uncertainties in the latter, make the above method of determining α grossly uncertain. While I will use $\alpha=1/10$ days in subsequent calculations, I do so only because this value is consistent with independent estimates.

7. Symmetric easterly waves: $n=1$

In Maruyama's (1967) analysis of easterly internal waves in the equatorial stratosphere, the discussion emphasized the antisymmetric mixed Rossby-gravity mode ($n=0$). However, such a mode is associated with zero amplitude zonal wind and temperature oscillations at the equator, and it is clear from Maruyama's analysis that oscillations in these fields did exist implying the existence of symmetric waves of similar phase speed and zonal wavenumber. Similarly, the existence of oscillations in northerly velocity at the equator unambiguously demonstrated the existence of antisymmetric waves.[3] Symmetric waves correspond to $n=1, 3, \ldots$. The Kelvin wave ($n=-1$) exists only as a westerly wave. Although no detailed observational analysis of symmetric stratospheric easterly waves is available, a detailed study of theoretical results for $n=1$ should be suggestive of what is going on. Such a study will also provide a concrete example of the kind of solutions obtained earlier in this paper.

I shall investigate the behavior of waves with $n=1$ whose phase speeds and zonal wavenumbers are the same as those of the Yanai waves studied in Part I, i.e.,

$$\left.\begin{aligned} c&=\frac{\omega}{k}=-23 \text{ m sec}^{-1} \\ k&=0.63\times10^{-3} \text{ km}^{-1} \end{aligned}\right\} \tag{95}$$

For $\bar{u}$, I take the distribution U_1 in Fig. 1 which is taken from Maruyama (1967). With the above choice of parameters, there are two solutions corresponding to Rossby and gravity waves [*viz.* Eq. (30)]. The details of the solutions are given in Section 4. The solutions for individual fields are multiplied by the damping factor given by (90). In addition, each field has been multiplied by a factor $e^{z/2H}$ (where I took $H=6$ km) corresponding to the effect of a non-Boussinesq atmosphere. In order for the reader to better understand the results I shall write the solutions in the following form:

$$u_0^{(1)}=u(z)U(z,y)\times\exp\left(-\int^z gdz\right)\exp\left(i\int^z fdz\right)\exp(z/2H), \tag{96}$$

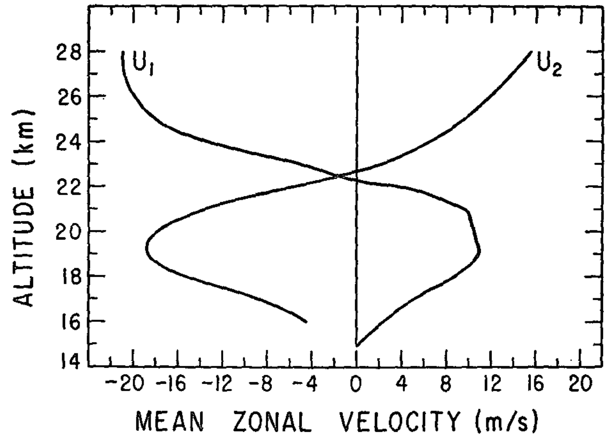

FIG. 1. Vertical distributions of mean zonal wind over the equator: U_1 is taken from Maruyama (1967) while U_2 is taken from Kousky and Wallace (1971).

where

$$u(z)=\frac{A_0}{\beta l_0^2}, \tag{97}$$

[*viz.* Eqs. (84) and (84a)] and

$$U(z,y)=\{2q_0H_0+H_2(\xi)\}e^{-\xi^2/2}, \tag{98}$$

where q_0 is a function of z and

$$\xi=\frac{y}{l_0}, \tag{99}$$

where l_0 is also a function of z [*viz.* Eqs. (31), (24) and (30)]. Similar breakdowns for $v_0^{(1)}$ and $\Phi_0^{(1)}$ are readily made, wherein

$$v_0^{(1)}=v(z)V(z,y)\exp\left(-\int_0^z gdz\right)\times\exp\left(i\int^z fdz\right)\exp(z/2H), \tag{100}$$

$$\Phi_0^{(1)}=\varphi(z)\Phi(z,y)\exp\left(-\int^z gdz\right)\times\exp\left(i\int^z fdz\right)\exp(z/2H). \tag{101}$$

In Fig. 2, the height distributions of $u(z)$ for both gravity waves and Rossby waves are shown as are the corresponding quantities for $v_0^{(1)}$ and $\Phi_0^{(1)}$, namely $v(z)$ and $\varphi(z)$. In Fig. 3 the height variations of $\exp(-\int_{16}^z gdz)$ and $\exp[(z-16)/12]\exp(-\int_{16}^z gdz')$ are shown for both gravity and Rossby waves. In Fig. 4 the variation of q_0 with height for both gravity and Rossby waves is shown, and Figs. 5–8 show the functions $U(z,y)$ and $\Phi(z,y)$ as functions of ξ for different

[3] One cannot overestimate the importance of data from equatorial stations like Kapingamarangi (01°02′N) where symmetric and antisymmetric waves can be clearly disentangled; this is no longer possible at stations away from the equator where each field includes contributions from both types of waves.

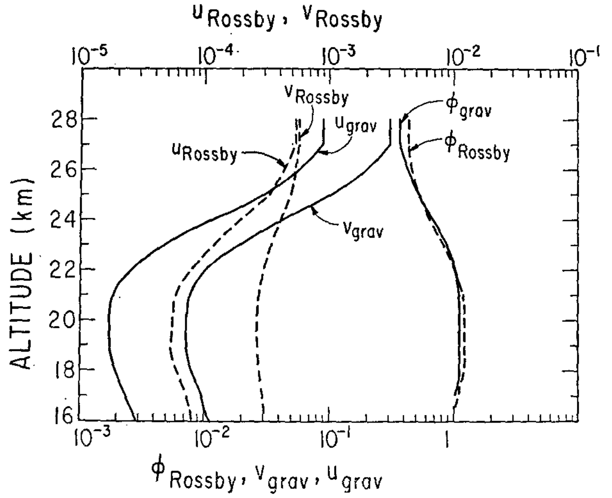

FIG. 2. Distributions of $u(z)$, $v(z)$, and $\varphi(z)$ [quantities defined by Eqs. (96), (100) and (101)] for $n=1$ gravity and Rossby waves when $\bar{u}$ is given by U_1 in Fig. 1.

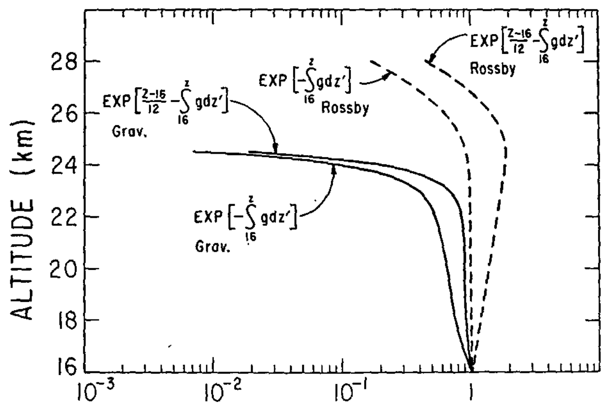

FIG. 3. Distribution of the dissipative damping factor

$$\exp\left[-\int_{16}^{z} g dz'\right] \quad \text{and of} \quad \exp\left[\frac{(z-16)}{12}-\int_{16}^{z} g dz'\right]$$

for $n=1$ gravity and Rossby waves.

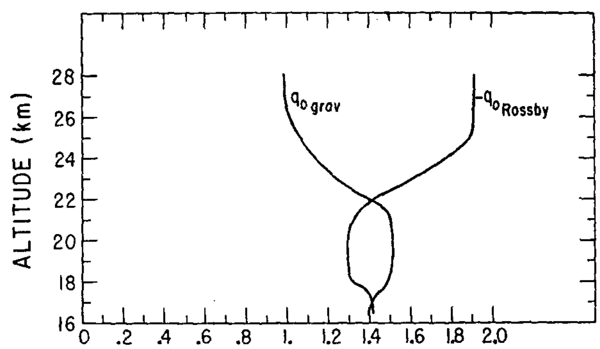

FIG. 4. The vertical distribution of the shape factor q_0 for $n=1$ gravity and Rossby waves.

values of q_0 [$V(z,y)$ is independent of q_0; *viz.* Eq. (83)]. Of course ξ is a scaled variable, and in Fig. 9 the height variation of the scaling factor l_0 is shown. The information in Figs. 2–9 is needed to determine the height variation of any field's amplitude at any latitude. As an example I show in Fig. 10 the height distributions of the amplitude of the zonal wind oscillation for both the $n=1$ Rossby and gravity waves at the equator. Also shown is the height distribution of the amplitude of the southerly velocity oscillation at the equator for a Yanai wave. All values have been normalized to 1 at 16 km. I shall discuss these results in some detail in the next

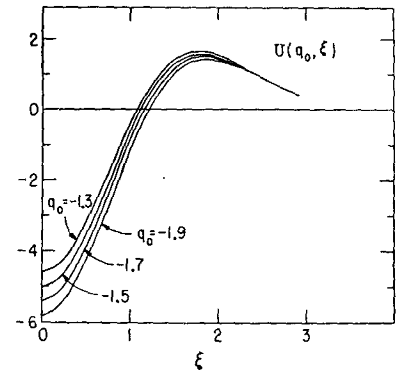

FIG. 5. $U(z,y)$ [*viz.* Eqs. (96) and (99)] as a function of scaled northward distance ξ for different values of q_0 obtained for an $n=1$ Rossby wave.

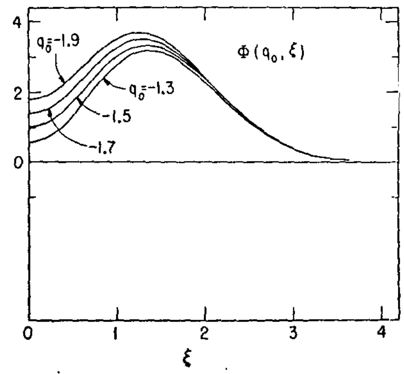

FIG. 6. $\Phi(z,y)$ [*viz.* Eq. (101)] as a function of scaled northward distance ξ for different values of q_0 obtained for an $n=1$ Rossby wave.

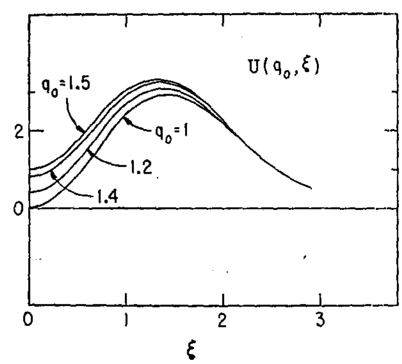

FIG. 7. Same as Fig. 5, but for values of q_0 obtained for an $n=1$ gravity wave.

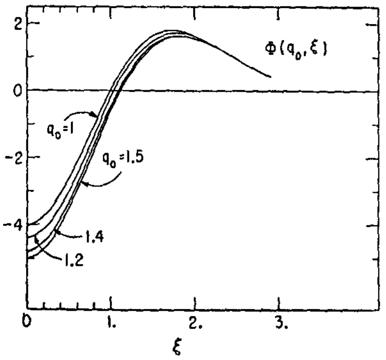

FIG. 8. Same as Fig. 6, but for values of q_0 obtained for an $n=1$ gravity wave.

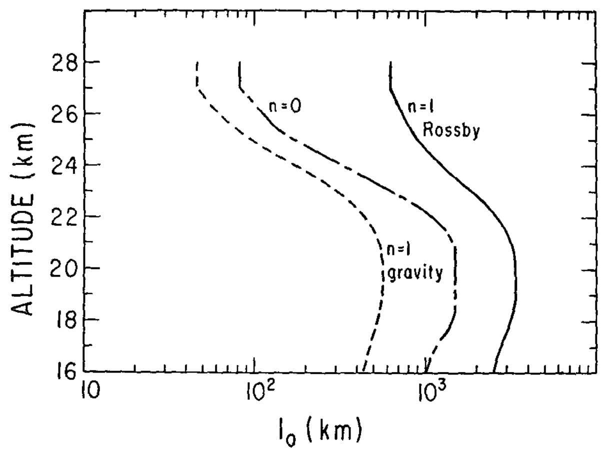

FIG. 9. The meridional scaling factor n_0 as a function of height for $n=1$ Rossby and gravity waves.

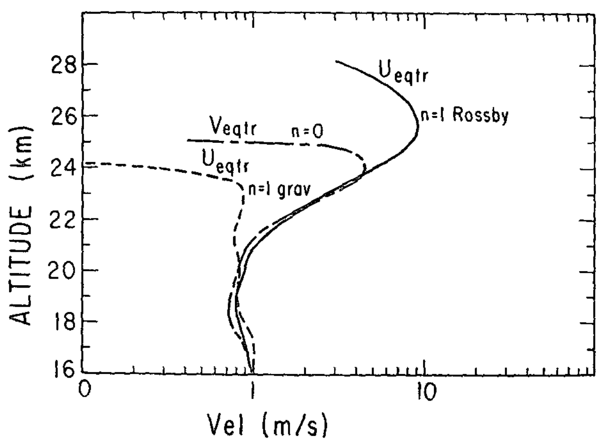

FIG. 10. Amplitude of zonal wind oscillation at the equator for $n=1$ Rossby (solid dashes) and gravity (short dashes) waves. Also shown is the amplitude of the southerly velocity oscillation at the equator for the $n=0$ Yanai wave (short-long dashes).

section. Not only is amplitude varying with height; local vertical wavelength ($2\pi/\text{Real} f_0$) is also varying as is shown in Fig. 11. Finally, in Fig. 12 I show the vertical distribution of $\langle \bar{F}_m \rangle$ (normalized to 1 at 16 km) for the $n=1$ waves and the Yanai wave.

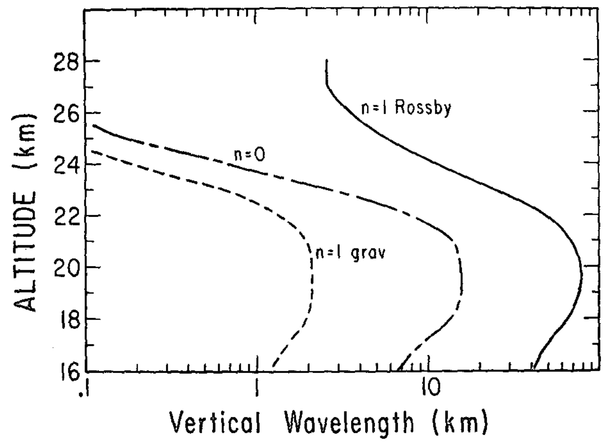

FIG. 11. Distributions with height of the local vertical wavelength ($2\pi/f_0$) for $n=1$ Rossby and gravity waves and for the $n=0$ Yanai wave.

8. Discussion of symmetric easterly waves

Table 1 shows the amplitude of the observed zonal wind oscillation at various heights over Kapingamarangi. For comparison I also show the southerly wind amplitudes. The values are taken from Maruyama (1967). The mean wind at the time the data were taken corresponds to U_1 in Fig. 1. One may reasonably ask how the symmetric modes described in Section 7 account for these admittedly uncertain observations. Clearly, in the absence of more data, no unique answer is possible. However, using Fig. 10, some conjectures are possible:

1) The Rossby mode cannot account for the observed values at 100 and 50 mb. If it did we would have to have much larger amplitudes at 25 mb (~25 m sec^{-1}) than anyone has observed.

2) The gravity mode might plausibly explain the observed amplitudes at 100 and 25 mb since its amplitude over the equator is almost constant with height

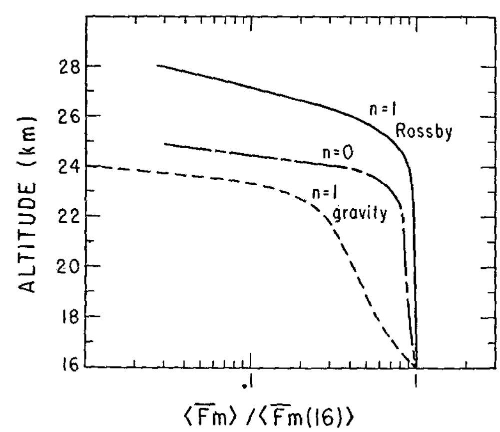

FIG. 12. Distributions of the latitudinally integrated vertical flux of zonal momentum $\langle \bar{F}_m \rangle$ (normalized to its value at $z=16$ km) for $n=1$ Rossby and gravity waves and the $n=0$ Yanai wave.

TABLE 1. Observed oscillatory wind amplitudes over Kapingamarangi (after Maruyama, 1967).

p (mb)	z (km)	U (m sec^{-1})	V (m sec^{-1})
100	16.5	3.4±1.4	3. ±1
50	21	2.6±0.8	2.4±0.9
25	25	2.6±1.2	2.5±1.2

from 16–23 km (However, its vertical wavelength, 1–2 km (*viz.* Fig. 11) is so short as to interfere with the accurate measurement of this mode). Above 23 km the amplitude decays rapidly, and it appears impossible for the gravity mode to explain the observed amplitude at 25 mb.

3) The observed amplitude at the equator might be accounted for by the sum of a gravity mode with an amplitude of about 2.2 m sec^{-1} over the equator at 16 km and a Rossby mode with an amplitude of about 0.38 m sec^{-1} over the equator at 16 km. For the Rossby mode this implies relatively weak forcing near 30° latitude (*viz.* Figs. 6 and 9). For the gravity mode one is faced with the problem that an amplitude of 2.2 m sec^{-1} at the equator implies an amplitude of about 6 m sec^{-1} near 7° latitude (*viz.* Figs. 7 and 9). The latter value is greater than Maruyama reports for Kusaie (05°20′N); however, as mentioned earlier, away from the equator there may be some cancellation by symmetric and antisymmetric modes. In favor of the importance of the Rossby wave near 25 km is the fact (*viz.* Fig. 11) that its wavelength at 25 km is the same as that of the Yanai wave some kilometers lower.

Before ending this discussion of internal easterly waves, I would like to emphasize that each of the modes discussed is effective in carrying momentum into the stratosphere. As we see in Fig. 12 the $n=1$ Rossby wave penetrates further than the Yanai wave, and the Yanai wave penetrates further than the $n=1$ gravity wave. Nevertheless, all the modes are absorbed below an almost critical level while even the gravity wave is able to carry appreciable momentum up to almost 23 km. However, the momentum carried by the Rossby and Yanai waves is deposited in rather thin regions while the momentum carried by the gravity wave is broadly deposited over the region 16–23 km. Finally, although the momentum flux due to the Rossby wave is distributed over a broad range of latitudes at the 100-mb level, by the time the Rossby wave reaches the levels at which it is strongly attenuated, its extent is largely confined to the tropics.

9. Westerly waves

In contrast to the situation for easterly waves, the main observed equatorial westerly waves are symmetric, and there is little evidence of a comparable antisymmetric component (Kousky and Wallace, 1971). Moreover, there are no westerly Rossby waves [*viz.* Section 3, especially Eq. (30a)], and as we have seen in Section 7 long-period equatorial gravity waves are (with the exception of the Kelvin wave) rapidly damped. Thus, the identification of the observed wave with Kelvin waves remains likely despite certain problems with this identification discussed in Part I; in consequence there is little need for a detailed discussion of higher order westerly modes similar to that presented for easterly modes. I merely show, in Fig. 13, some results for $\langle \bar{F}_m \rangle$ vs z for a Kelvin wave, a Yanai wave, and an $n=1$ gravity wave. For each of these calculations I have used U_2 as given in Fig. 1. In addition I have used $c=18$ m sec^{-1}, $s=-2$, corresponding to $k=3.12\times10^{-4}$ km^{-1}, and an unDoppler shifter period of about 13 days. We see in Fig. 13 that for the above choice of $\bar{u}$, c, and s only the Kelvin wave can transmit zonal momentum upward from the lower boundary. The gravest antisymmetric mode (a westerly Yanai wave) barely makes it to 17.5 km while the $n=1$ wave hardly escapes the boundary. This behavior is associated with the vertical

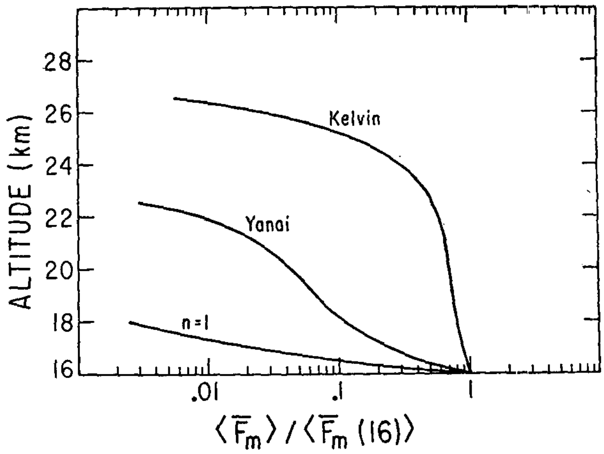

FIG. 13. Same as Fig. 12, but for westerly Kelvin, Yanai, and $n=1$ gravity waves.

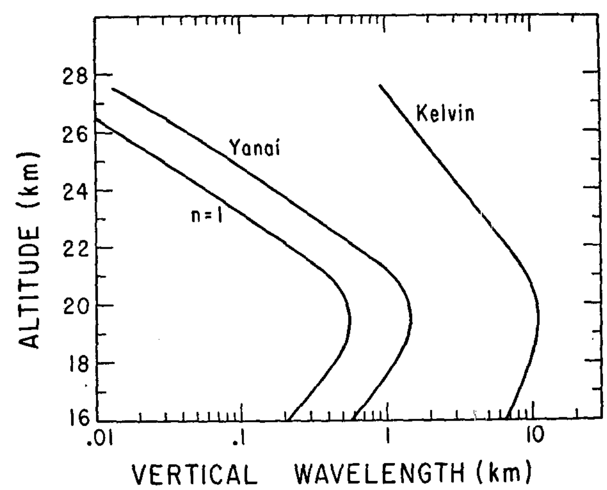

FIG. 14. Same as Fig. 11 but for westerly Kelvin, and $n=1$ gravity waves.

wavelengths associated with the various modes. These are shown in Fig. 14. While the local vertical wavelength of the Kelvin wave exceeds 10 km over a significant range of altitudes, the wavelengths of the other modes barely exceed 1 km.

10. Propagating diurnal tide

As pointed out in Lindzen (1967b) the main diurnal propagating tidal mode corresponds, in the present notation, to an $n=1$ gravity mode for which $\sigma=2\pi/1$ day and $s=1$. These values yield $c=-465$ m sec^{-1}. We are, with our present formulae, in a position to evaluate the effects of wind on this mode. In general this mode is primarily excited by insolation absorbed by water vapor primarily below 18 km [a complete discussion of atmospheric tidal modes is given in Chapman and Lindzen (1970)]. Above this height, tropical winds vary between ± 30 m sec^{-1} below 60 km (Reed, 1964, 1966). Above this height little is known about mean zonal winds; conceivably they might be of the order of ± 50 m sec^{-1} below 100 km. In general, since these speeds are much smaller than c we may expect the effect of mean winds on the propagating diurnal tide to be small. However, certain quantitative aspects of the effect are worth noting.

First, I will describe what is *not* affected by winds of the stated magnitude:

1) The quantity q_0 remains close to 1.4.

2) The quantity $\varphi(z)$ [*viz.* Eq. (101)] remains virtually constant.

3) The effect of dissipation as given by the factor $\exp(-\int^z g dz')$ in Eqs. (96), (100) and (101) is relatively independent of mean wind. The dissipative model described in Section 6 leads to a 15% reduction of amplitude (relative to dissipationless results) between 20 and 90 km, for both easterly and westerly winds. In both cases $g\approx 1/(450 \text{ km})$.

On the other hand, one may use Eqs. (24), (30), (77), (83) and (84) to show that the following effects do occur:

1) Relative to a local vertical wavelength of about 23.5 km without wind, an easterly wind of 30 m sec^{-1} will decrease this to 20 km while a westerly wind of 30 m sec^{-1} will increase this to 27 km. Due to long-period variations of mean wind, westerly and easterly regimes do exist over the region 20–50 km implying corresponding variations in phase for the propagating diurnal mode on the order of 90° at 60 km.

2) Local changes in mean zonal wind of ± 30 m sec^{-1} will lead to local changes of the quantities $u(z)$ and $v(z)$ in Eqs. (96) and (100) of $\pm 12\%$, the larger values being associated with easterly winds.

3) Local changes in mean zonal wind of ± 30 m sec^{-1} will lead to local changes of meridional scale l_0 of $\pm 8\%$, with the largest scale corresponding to westerly wind. Such changes of scale can lead to large changes in tidal amplitude in the vicinity of zeros and extrema of tidal modes (*viz.* Figs. 7 and 8). In the absence of wind, $l_0\approx 1900$ km.

My main purpose in presenting these results for the propagating diurnal tide is to offer a quantitative caveat against the precise comparison of observations with theoretical results from classical tidal theory where the effect of winds is ignored.

11. Conclusion

The bulk of my conclusion is simply a repetition of the introduction. However, I would like to add that the asymptotic solutions have permitted the solution of problems whose short scales would have precluded the economic application of finite-difference numerical techniques. As a result, the possibility arises that Greens functions based on the present asymptotic solutions might provide a rapid and economical means for evaluating the atmosphere's response to various forcing mechanisms. The possibility is currently being explored as a possible alternative to the numerical approach of Holton (1972).

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