NOTES AND CORRESPONDENCE

A Note on the Limits of Rossby Wave Amplitudes

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ABSTRACT

The constraints imposed by conservation of potential vorticity and hydrodynamic stability on the amplitude of Rossby waves are investigated.

1. Introduction

Quasi-geostrophic dynamics, as usually formulated, centers on the conservation of pseudo potential vorticity (hereafter simply potential vorticity or PV) on isobaric surfaces (Charney and Stern, 1962; Bretherton, 1966). The potential vorticity on a β-plane is given by

$$q = \tilde{\nabla}^2\psi + f, \tag{1}$$

where $f = f_0 + \beta y$ and

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{1}{\rho N^2} f_0{}^2 \frac{\partial}{\partial z} \rho \frac{\partial}{\partial z},$$

ψ is the quasi-geostrophic stream function and x, y, and z are eastward, northward and upward coordinates, respectively; z is $H \ln(p_0/p)$ where H is a scale height and p_0 is a reference pressure; ρ is a basic state density, $\rho = \rho_0 \exp(-z/H)$, where ρ_0 is a constant; N is the buoyancy frequency assumed here to be constant.

The purpose of this note is to investigate some simple constraints that conservation of PV imposes on internal Rossby waves. The conservation of PV can be written as

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \mathbf{v}_g \cdot \nabla q = S, \tag{2}$$

where S is the source or sink of PV. For the β-plane $\mathbf{v}_g$ is simply the geostrophic wind.

In this note we will examine some of the limitations that conservation of potential vorticity imposes on the dynamics of the stratosphere. Another possible constraint imposed by hydrodynamic stability will also be discussed. The quantitative limitations these constraints put on the amplitudes of internal Rossby waves are assessed and compared with observations of the lower stratosphere. Incidentally, we show that a question posed by Charney in 1960 (Charney and Drazin, 1961) as to whether Rossby waves can produce a terrestrial corona can be answered without reference to the dispersion properties or dissipation mechanisms of Rossby waves.

2. Constraints on the amplitude of internal Rossby waves for stable or unstable basic flows

The study of internal Rossby waves (and other quasi-geostrophic perturbations including baroclinic instabilities) is almost always based on the linearized version of (2) solved for perturbations about a basic zonal flow $\bar{u}(y, z)$. The nature of these waves is discussed at length by Charney and Drazin (1961), Dickinson (1968), Matsuno (1970), Simmons (1974), Schoeberl *et al.* (1979) and others. If vertical propagation of the Rossby waves is possible then the amplitude of the geopotential height perturbations for steady waves increases with altitude approximately as $e^{z/2H}$. This behavior reflects the fact that wave action is conserved as the wave propagates. However, wave potential vorticity is drawn from the basic state at each level, and linear theory assumes that the basic state is an infinite source. For waves where the perturbation PV is small, no difficulty arises.

For finite amplitude waves, (2) permits us to set some bounds on the amplitude of wave perturbations. These bounds are dictated by the fact that if there is no source of PV on a given isobaric surface then

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the PV of a wave cannot exceed the PV available from the basic state on the same surface. This result is obvious from (2) and leads to the following constraint.

The perturbation PV, q', at any point cannot be greater than the maximum or be smaller than the minimum PV found in the basic state on the isobaric surface passing through that point unless a source of PV is present. Since the development of a wave on an isobaric surface according to (2) can be viewed simply as the rearrangement of the isopleths of PV on that surface, this condition follows directly (viz Dickinson, 1968).

A special case arises when the basic state PV, $\bar{q}$, is everywhere positive. This is generally the case; indeed if $\bar{q}$ changes sign, the basic state is inertially unstable. If $q > 0$ everywhere on the isobaric surface then the amplitude of the perturbation at any point, $|q'|$, cannot exceed $\bar{q}$ at the same point. If it did, the total PV at that point would turn negative during the wave's negative phase. It is this special case that we will employ in Section 3.

3. Quantitative estimates for the winter stratosphere, and the impossibility of a terrestrial corona due to Rossby waves

The purpose of this section is to assess the amplitudes of planetary waves permitted in the stratosphere using the condition described in Section 2. We have evaluated $\bar{q}$ for the zonal wind and temperature profiles given in Lindzen and Hong (1974). We find the approximation $\bar{q} \approx f = 2\Omega \sin\theta$ adequate for present purposes. Fig. 1 shows the amplitude and phase of the climatological planetary zonal harmonic one ($s = 1$) from van Loon *et al.* (1973). Although $s = 1$ suggests a large zonal scale, Fig. 1 shows that the meridional scales are shorter, O(1600 km), and $\nabla^2\psi' \approx \partial^2\psi'/\partial y^2$. The contributions of the vertical variations to (1) also seem relatively small in the lower stratosphere.

For present illustrative purposes a rough evaluation of our condition should suffice. Thus, for $q' \approx \tilde{\nabla}^2\psi' \approx 1/f\nabla^2\Phi$, we will take

$$q' \approx gZ'/fL^2, \qquad (3)$$

where $g = 9.8$ m s^{-2}, Z' = the amplitude of the height perturbation defined as $gZ' = \Phi'$, L = the characteristic meridional scale (1600 km). Requiring that $|q'| < \bar{q} \approx f$ yields

$$Z' < f^2L^2/g = \frac{4\Omega^2L^2}{g}\sin^2\theta = 5500 \text{ m} \sin^2\theta. \qquad (4)$$

Clearly, the results in Fig. 1 satisfy (4) by a large amount. Thus it is not clear that our constraint forms a practically usable limit on wave amplitudes. We shall return to this matter in Section 4.

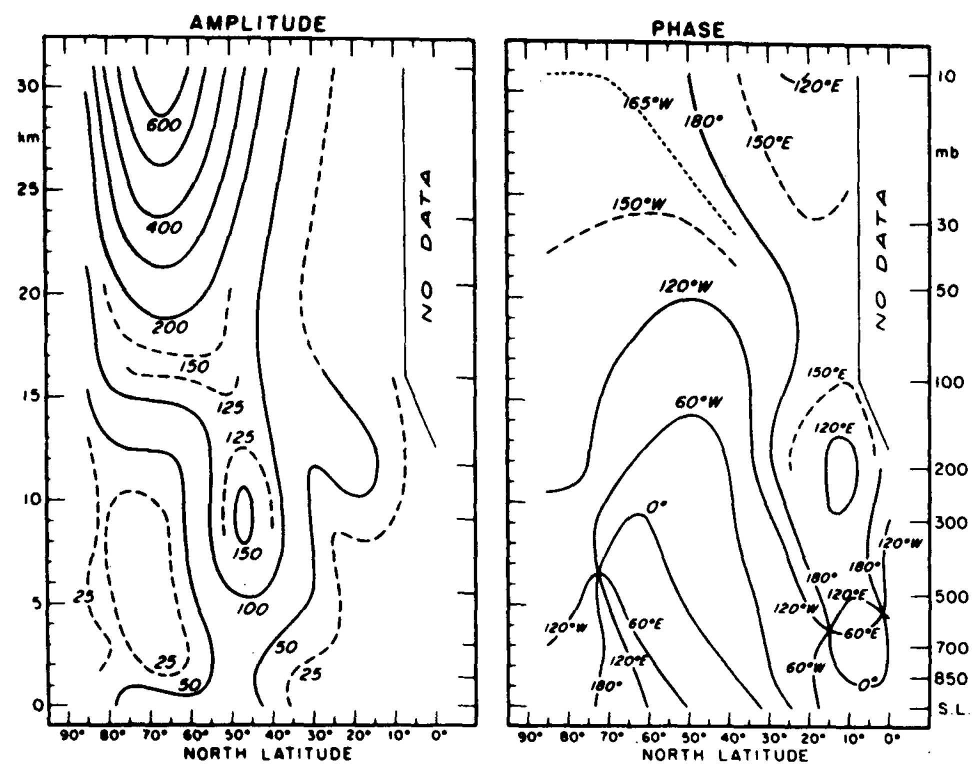

FIG. 1. The climatological amplitude and phase of planetary wave one (zonal harmonic one) in January from van Loon *et al.* (1973).

Inequality (4) can, however, be used to answer in the negative the question of whether vertically propagating Rossby waves can produce a terrestrial corona. This question was the motivation for the seminal paper by Charney and Drazin (1961) on the vertical propagation of Rossby waves. It was known that vertically propagating Rossby waves would grow approximately as $e^{z/(2H)}$ so as to conserve wave action. Thus waves which would be thermodynamically inconsequential in the upper troposphere could become overwhelmingly important to the heat budget of the thermosphere where the large molecular viscosity and thermal conductivity would lead to the deposition of Rossby wave energy.

Charney and Drazin (1961) used linear theory to examine the dispersive properties of these waves including those properties which would prevent vertical propagation. Their study has been amplified by subsequent authors and the results have explained many of the salient features of stationary waves observed in the stratosphere.

Ironically, however, the answer to the original question posed by Charney and Drazin (1961) can be provided without reference to the propagation properties of Rossby waves. The limit given by (4), while larger than amplitudes shown in Fig. 1, is still two orders of magnitude smaller than amplitudes expected in the lower thermosphere on the basis of $e^{z/(2H)}$ growth. Hence, the limit given by (4) represents an energy density four orders of magnitude smaller than that which Charney and Drazin (1961) suggested could produce temperatures $\sim$ 100 000 K.

4. Further remarks

The amplitudes in Fig. 1 are apparently limited by other processes than the availability of PV. The maximum amplitude in Fig. 1 is only 600 m. In another more recent analysis by Barnett (1980) where satellite radiance data was analyzed to produce meridional sections of wavenumber 1 height fields similar to those in Fig. 1 (but for an average over Dec, 1975, Jan, Feb, 1976 rather than over all Januaries for 7 years), the maximum amplitude was again found to be 600 m. In this case, however, the maximum was found near 50 km; at 30 km amplitudes were only 300 m. Barnett's analysis extends to 80 km rather than to only 30 km as in Fig. 1.

The maximum amplitude found in the above mentioned studies suggests the possibility that another constraint, besides the availability of potential vorticity, may be operating: namely, that the contribution of the wave to q_y, $|q'_y|$, should not exceed $\bar{q}_y$. Along the lines of Section 4, we may approximate $\bar{q}_y$ with β ($=df/dy$), and

$$q'_y \approx qZ'/fL^3.$$

Requiring $|q'_y| < \bar{q}_y \approx \beta$ then yields

$$Z' \lesssim f\beta L^3/g = \frac{4\Omega^2 L^3 \sin\theta \cos\theta}{ga} \approx 700 \text{ m} \sin 2\theta, \quad (5)$$

which is very nearly what is observed.

There is, as yet, no rigorous basis for this constraint, but it is not implausible. The restoring force for quasi-geostrophic waves results from the meridional variation of q. The existence of waves for which $|q'_y| > \bar{q}_y$ would locally eliminate this restoring force. Moreover, if $|q'_y| > \bar{q}_y$, then there will be surfaces where q_y changes sign, and this is generally associated with some mixture of barotropic and baroclinic instability (Charney and Stern, 1962; Lindzen and Tung, 1978). Unfortunately, the stability analysis of complicated three-dimensional fields is notoriously difficult (Niehaus, 1981). Moreover, early in the history of hydrodynamic stability analyses, it was already noted that normal mode results may not even be relevant. Orr (1907) studied flows which lacked normal mode instabilities and for which all initial perturbations eventually decayed algebraically. He nevertheless discovered that many initial perturbations could grow rapidly to large amplitude before decaying. If such perturbations led to turbulence, the long time behavior would not be of consequence. Orr's work is reviewed and extended in the recent work of Farrell (1982). One last and perhaps most likely possibility is that (5), itself, does follow from conservation of potential vorticity. Recall that (4) is a very conservative statement of the condition. More generally, we expect that

$$q = \bar{q} + q' \approx f + q' \quad (6)$$

not be different from a value of $\bar{q} \approx f$ reached someplace on the relevant isobaric surface. If one focusses on a wave of meridional extent $2L$ centered at $y = y_0$, then

$$f_0 - \beta L < f < f_0 + \beta L. \quad (7)$$

At y_0,

$$q \approx f_0 + q',$$

and (5) follows immediately. It is, however, unclear as to whether one can assume fluid excursions are indeed bounded by L.

The most serious objection to (5) arises from the recent analysis of Labitzke (1981) for winter 1979 (and supported by the more recent analyses of satellite data by T. Palmer, pers. comm., 1981). This analysis suggests that perturbation heights generally exceed 1000 m near 50 km and occasionally reach 2500 m! The absence of such amplitudes in Barnett's analysis strongly suggests that the disturbances seen by Labitzke do not represent stationary waves. Indeed, assuming that the analysis is correct, it seems that the disturbances may not even be quasi-geostrophic. It is easily shown that height perturbations of 2500 m will be associated with geostrophic zonal wind "perturbations" amplitudes of over 100 m s^{-1}

and meridional wind oscillations almost as large. The latter are almost never reported at 50 km—except perhaps in connection with gravity waves. The question of the accuracy of such analyses is crucial for work on stratospheric dynamics. The constraint on $\bar{q}_y$ suggested above, if it can be proven to be relevant, may prove useful in this regard. In this connection, however, it should be stressed that the numbers used in expressions (4) and (5) are only rough estimates, and that the limiting amplitudes depend strongly on horizontal scale, L. In practice, the formulas rather than the illustrative numbers should be used.

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