Quasi-geostrophic Wave-CISK in an Unbounded Baroclinic Shear

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ABSTRACT

In this study, the free-shear problem, a minimal version of baroclinic, quasi-geostrophic wave-CISK, is analyzed. The basic state consists of a zonal flow, unbounded above and below, with constant vertical shear and Brunt–Väisälä frequency and zero meridional gradient of the potential vorticity; and convective heating is parameterized in terms of the convergence below an arbitrary level. Because of the sensitivity to the vertical distribution of the parameterized heating typical of wave-CISK models, a simple thermodynamic constraint on the heating profile is used to broadly identify appropriate parameter regimes. The unstable waves in the free-shear problem grow rapidly and share many structural characteristics with dry baroclinic waves, although the dynamical process associated with dry baroclinic instability is absent; consideration of the potential vorticity dynamics of the unstable modes illustrates how heating may act as a dynamical surrogate for potential vorticity gradients. Although highly idealized, the free-shear problem also explains much of the behavior of more general wave-CISK models.

1. Introduction

There has been increasing recognition over the past 30 years that the diabatic effects of moist convection can play an important role in extratropical cyclogenesis. Beginning with Sanders and Gyakum (1980), observational studies based on both case studies and climatology have suggested that the most rapidly deepening synoptic-scale lows, while of primarily baroclinic origin, are significantly influenced by a variety of diabatic processes, particularly those associated with moist convection. Numerical simulations of such storms have supported this view (e.g., Kuo and Reed 1988), since the simulated deepening rates with the moist convective parameterization and resolvable-scale latent heat release turned off are in general about half of that in the full simulations. Other, smaller-scale cyclones such as polar lows and comma clouds also appear to be associated with both baroclinic and moist processes (Reed 1979), although there is as yet little consensus on the relative contribution of each process (Rasmussen and Lystad 1987).

Wave-CISK is one approach to understanding how the presence of organized moist convection can affect baroclinic waves. CISK (conditional instability of the second kind) originated as an explanation of tropical cyclones (Charney and Eliassen 1964) and assumes that there is a positive feedback between the heating due to moist convection and the large-scale circulation that organizes the convection. In this early work, the apparent heating due to moist convection was assumed to be proportional to the large-scale frictional convergence at low levels. Later, in studies of tropical easterly waves, the effects of moist convection were parameterized in terms of the convergence by an inviscid large-scale wave, a method that has come to be known as wave-CISK (see Davies 1979, for a review). These ideas have been subsequently applied to midlatitude cyclones; using the quasi-geostrophic equations and convective parameterizations based on large-scale convergence, several authors have considered the effects of cumulus heating on Rossby waves in a baroclinic basic state (Mak 1984; Moorthi and Arakawa 1985; Wang and Barcilon 1986; Bannon 1986; Craig and Cho 1988). Even pure, frictionally driven CISK in a uniform basic state has been proposed as the mechanism that produces polar lows (Rasmussen 1979).

Efforts to characterize the behavior of observed or numerically simulated extratropical cyclones in terms of baroclinic wave-CISK models have met with mixed results. The models used in these efforts have consistently shown that parameterized heating can significantly increase the growth rate and alter the structure of dry baroclinic waves. The results obtained, however, are very sensitive to the intensity and vertical distribution of the heating (Wang and Barcilon 1986) and

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these quantities are in general not determined by the convective parameterization, but are instead specified empirically. In addition, little has been revealed about the fundamental mechanisms by which the heating can alter the dynamics from those of a dry baroclinic Rossby wave, beyond suggestions that the heating decreases the effective static stability seen by the wave.

The intention of this study is to analyze the basic dynamical processes that occur when disturbances are free to interact with both a baroclinic zonal flow and parameterized heating. The parameterization of the heating will not be justified, beyond showing that it satisfies rudimentary budgets of heat and moisture; the convective parameterization is of interest primarily as one potential description of the interaction between the wave and the heating. However, to reduce the arbitrariness in the choice of the heating profile, we propose a thermodynamic constraint relating the depth of the convection to the basic state specific humidity and stratification. To aid the analysis, we introduce a highly simplified version of baroclinic wave-CISK (termed the free-shear problem), which retains both the basic state shear and the heating, but which lacks a basic state potential vorticity gradient and upper and lower boundaries, and so dispenses with the dynamical process associated with dry baroclinic instability. (Throughout this paper, potential vorticity will refer to the quasi-geostrophic analog to Ertel's potential vorticity.) The simplicity of this model allows one to effectively explore and understand its possible behaviors and does not appear to prohibit its application to more general situations.

The free-shear problem reproduces much of the characteristic behavior of previous, more complicated baroclinic wave-CISK models. In particular, in the parameter regime specified by the thermodynamic constraint on the heating profile, the dispersion relation and structure of the unstable modes resemble those of dry baroclinic waves, although the growth of modes in the free-shear problem results primarily from the generation of perturbation available potential energy by the heating. The dynamics and parameter dependencies of the instabilities in the free-shear problem may be understood in terms of the relation between the vertical velocity and various configurations of perturbation potential vorticity. In fact, this analysis of the modal potential vorticity dynamics leads to a fundamental (if intuitively obvious) conclusion that parameterized heating may substitute for basic state potential vorticity gradients, since a disturbance can then create and modify potential vorticity perturbations even in the absence of basic state gradients of potential vorticity.

2. Formulation

a. Basic equations

We linearized the quasi-geostrophic equations on a β-plane about a baroclinic basic state of y-independent zonal flow $U = U(z)$ and static stability $N^2 = N^2(z)$. Following Stevens and Lindzen (1978), the basic state includes a mean cumulus activity and precipitation; radiative effects are assumed to balance the mean heating and maintain a steady large-scale basic state, while surface fluxes of sensible and latent heat produce the potentially buoyant air at low levels necessary to maintain the cumulus. Although somewhat contrived at midlatitudes, this basic state admits a straightforward linearization in which both positive and negative perturbation heating are well defined. In order to parameterize the perturbation heating in terms of the large-scale motion, we assume that the deviations from the mean convective state are induced by the wave through the modulation of the moisture available to the clouds. Specifically, the perturbation heating will be proportional to that portion of the vertically integrated moisture convergence by the wave that involves potentially buoyant air. Thus, if the basic state specific humidity q is well mixed with uniform value q_0 below some height z_q and is negligible above z_q and air below z_q is assumed to be potentially buoyant, then the moisture convergence by the wave, and hence the perturbation heating, may be directly related to the vertical velocity at $z = z_q$; namely,

$$-\int_0^\infty \nabla\cdot(\rho q\mathbf{v})dz = q_0\rho(z_q)w(z_q) = q_0\rho_q w_q, \quad (1)$$

where the subscript q denotes evaluation at $z = z_q$.

Therefore the perturbation heating can be written as

$$\dot{Q} = \frac{\rho_0}{\rho_s}\frac{c_pT_0N^2}{g} w_qQG(z), \quad (2)$$

where $\dot{Q}$ is the perturbation heating rate in J kg^{-1} s^{-1}, $\rho_s(z)$ is the horizontally averaged density (assumed to have constant scale height $H = c_pT_0/g$), $\rho_0 = \rho_s(0)$, and T_0 is a mean tropospheric temperature. Q and $G(z)$ give the nondimensional amplitude and normalized vertical structure, respectively, of the perturbation heating and will be explicitly specified in the following subsection. Essentially similar parameterization schemes have been used, in various guises, in previous investigations of moist baroclinic instability (Moorthi and Arakawa 1985; Wang and Barcilon 1986; Craig and Cho 1988). Variations in the specification of Q and $G(z)$ will be considered in order to delineate specific sensitivities to details of the paramaterization.

The resulting perturbation potential vorticity equation, assuming that the perturbations vary as $e^{ikx} \times \cos(ly)$, is,

$$(U-c)\left\{\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s f^2}{N^2}\phi_z\right) - (k^2+l^2)\phi\right\} + \Pi_y\phi = \frac{f\rho_0}{ik\rho_s}w_qQ\frac{\partial G}{\partial z}, \quad (3)$$

where ϕ is the geostrophic streamfunction and Π_y is the basic state potential vorticity gradient. Note that only the vertical derivative of the heating appears in (3), since it is the variation of the heating with height that alters the separation of the isentropes and results in generation of potential vorticity.

In deriving the perturbation potential vorticity equation, our job was greatly simplified by the fact that the perturbation heating shares the wave's horizontal structure, being proportional to w_q as in (2). Thus, heating occurs where there is ascent at the top of the moist layer, while cooling occurs when $w_q < 0$. Alternatively, as in other applications of moisture convergence schemes to wave stability problems (e.g., Lindzen 1974; Bratseth 1985), one can begin with "conditional" heating that is nonzero only in regions of moist layer ascent. The heating is then expanded in a Fourier series and, although the problem is nonlinear as posed, only the term with the same wavenumber k as the wave is retained.

For future reference, the dimensionless form of (3) is

$$(\tilde{U}-\tilde{c})\left\{e^{\tilde{z}}\frac{\partial}{\partial \tilde{z}}\left(\frac{e^{-\tilde{z}}}{\tilde{N}^2}\tilde{\phi}_{\tilde{z}}\right)-(\tilde{k}^2+\tilde{l}^2)\tilde{\phi}\right\}+\tilde{\Pi}_{\tilde{y}}\tilde{\phi} = \frac{1}{i\tilde{k}}e^{\tilde{z}}\tilde{w}_q Q\frac{\partial G}{\partial \tilde{z}}, \quad (4)$$

where the nondimensional variables are

$$\left.\begin{aligned}\tilde{t} &= fmN_0^{-1}t\\ \tilde{z} &= H^{-1}z\\ (\tilde{k},\tilde{l}) &= N_0Hf^{-1}(k,l)\\ (\tilde{U},\tilde{c}) &= m^{-1}H^{-1}(U,c)\\ \tilde{\phi} &= fN_0^{-1}H^{-1}u_0^{-1}\phi\\ \tilde{w} &= N_0^2f^{-1}m^{-1}u_0^{-1}w,\end{aligned}\right\} \quad (5)$$

and m, N_0, and u_0 are scales for the basic state shear, basic state buoyancy frequency, and the perturbation horizontal velocity, respectively.

b. *Specification of Q and $G(z)$*

The physically allowable values of Q [for a given normalization of $G(z)$] are determined by the large-scale budgets of heat and moisture. If the effects of the wave on surface fluxes and the possibility of moisture storage in the clouds are ignored, and if the distribution of specific humidity outside the clouds is (approximately) steady, then the perturbation heat and moisture budgets are, respectively,

$$\int_0^\infty \rho\dot{Q}dz = LP, \quad (6)$$

$$-\int_0^\infty \nabla\cdot(\rho q\mathbf{v})dz = P, \quad (7)$$

where P is the perturbation precipitation.

Using (1) to rewrite (7) in terms of w_q and combining the result with (6) relates the nondimensional amplitude of the heating to the basic state parameters,

$$Q = \frac{\rho_q}{\rho_0}\frac{gLq_0}{c_pT_0N_0^2H}, \quad (8)$$

where the following normalization was used

$$\int_0^\infty G(\tilde{z})\tilde{N}^2(\tilde{z})d\tilde{z} = 1. \quad (9)$$

Physically, these budgets, and the assumptions made in deriving them, require that all moisture converged by the wave condenses and falls as rain, and that all moisture convergence involves air that is potentially buoyant. In addition, the wave must have relatively small effects on surface fluxes of latent and sensible heat; surface fluxes then contribute to the perturbation dynamics only by maintaining the basic state.

It remains to specify the vertical structure of the heating. Previously, $G(z)$ has been chosen a priori without reference to the underlying thermodynamics, and parameter dependencies have been determined by varying Q and $G(z)$ independently. Intuitively, however, one expects that changes in the basic state parameters, such as q_0 or N_0, will produce changes in both Q and $G(z)$. For instance, if the specific humidity in the moist layer is increased while the basic state stratification is held constant, the level of neutral buoyancy for parcels near the surface will increase and clouds will likely be deeper. The increase in Q must then be accompanied by an increase in the depth over which $G(z)$ is nonzero.

A constraint on the distribution of heating can be obtained by crudely estimating the depth of convection in terms of the basic state variables, following Stevens and Lindzen (1978, Appendix 1) and Lindzen (1988). For a potentially buoyant parcel, we assume, consistent with (6) and (7), that all moisture condenses and precipitates as the parcel rises; the dry static energy of the parcel, $s = c_pT + gz$, therefore changes by an amount $\Delta s = Lq_0$. Now suppose that the deep, precipitating clouds entrain very little, so that the buoyancy generated by the condensation is not degraded as the parcel rises. For parcels originating in the mixed layer and rising until their temperature is equal to that of the environment, the depth of the clouds, Δz, is then given implicitly by

$$\Delta s = \int_{z_q}^{z_q+\Delta z}\frac{\partial s}{\partial z}dz.$$

In terms of nondimensional quantities,

$$\frac{\rho_0}{\rho_q} Q = \int_{\tilde{z}_q}^{\tilde{z}_q+\Delta\tilde{z}} \tilde{N}^2(\tilde{z})d\tilde{z}, \qquad (10)$$

where the relation $\partial s/\partial z = c_pT_0N^2/g$ has been used.

While the above discussion does not fully describe how the vertical structure of the heating should depend on the basic state parameters, the thermodynamic constraint (10) still provides a general assessment of the physical consistency of a given choice of Q and $G(z)$, and indicates the correct dynamical regime for the instability problem. Since (10) is based on estimates of the height to which mixed layer parcels will rise by the complete condensation of water vapor, we take Δz to be the depth of the deepest cloud and, because the heating will be nonzero only where there is nonzero cloud mass flux, require $G(z) \neq 0$ for $0 < z - z_q < \Delta z$.

The budgets used to restrict Q and $G(z)$ are unquestionably idealized and represent a simplification of the full physics. Nevertheless, the constraint (10), by restricting the arbitrary choice of $G(z)$, insures that the resulting parameterization is thermodynamically consistent within this simplified model. An additional discussion of some processes that might alter the relation of Q and $G(z)$ appears in the Appendix.

3. The free-shear problem

For a general basic state and heating profile, the baroclinic wave-CISK system becomes intractable. Analytic solutions are not available, and the presence of potential vorticity gradients in the basic state (suitably generalized to include temperature gradients on horizontal boundaries) complicates the situation, by allowing instability independent of the heating. In order to gain more insight into the moist instability, a greatly simplified model was introduced. All variables in this section are dimensionless, and tildes will be dropped.

Basic state potential vorticity gradients may be eliminated by using the f-plane and Boussinesq approximations and assuming a basic state zonal flow, *unbounded* above and below, with constant vertical shear and constant Brunt–Väisälä frequency.[1] This is the basic state of the Eady problem (Eady 1949), but because of the lack of upper and lower boundaries, the flow is stable in the absence of heating. To simplify matters even further, one can take $z_q = 0$ and choose the heating profile to be uniform between $z = 0$ and $z = \Delta z$. The normalization (9) then gives

$$G(z) = \begin{cases} \Delta z^{-1} & \text{if } \; 0 < z < \Delta z \\ 0 & \text{otherwise.} \end{cases}$$

The geometry of the problem is shown in Fig. 1.

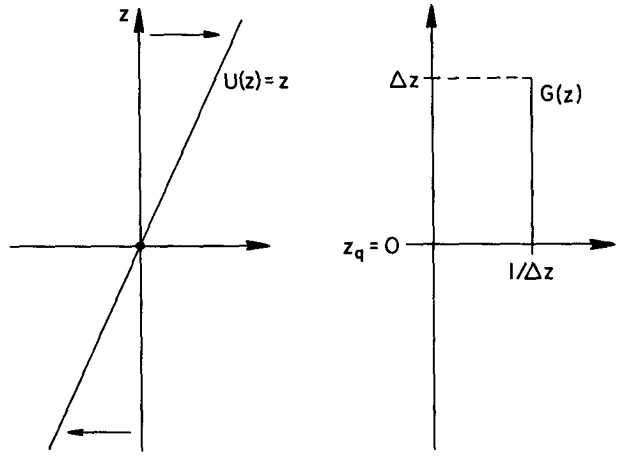

FIG. 1. The geometry of the free-shear problem, showing the basic state zonal wind and the normalized heating profile.

With these simplifications, (4) becomes

$$\phi_{zz} - k^2\phi = -\frac{w_qQ}{ik\Delta z}\left[\frac{\delta(z)}{c} + \frac{\delta(z-\Delta z)}{\Delta z - c}\right], \qquad (11)$$

where, for convenience, we consider y-independent perturbations. The δ-functions in the right-hand side of (11) arise through the differentiation of $G(z)$. In addition, the thermodynamic equation, which relates ϕ and w, is

$$ik[(z-c)\phi_z - \phi] + w = Qw_qG(z). \qquad (12)$$

Since the domain is unbounded, we require $\phi \to 0$ as $z \to \pm\infty$.

Under the Boussinesq approximation and the assumption that N^2 is constant, the thermodynamic constraint on the depth of the heating (10) reduces to $\Delta z = Q$. In this case, the diabatic term in the thermodynamic equation is equal to w_q, for $0 < z < \Delta z$.

The parameterized heating in this model clearly does not correspond to a real physical process, nor is the geometry realizable. Instead, the model includes by design nothing but a basic state shear and heating generated by vertical motion at $z = z_q$. The horizontal boundaries and basic state potential vorticity gradient are gone, removing all nontrivial modes not associated with the heating, and the heating is piecewise uniform, leading to tractable δ-function forcing in (11). The unbounded, free-shear problem is the minimal problem that allows perturbations to interact with both the shear and the parameterized heating.

a. *Dispersion relation and vertical structure for the unstable waves*

The Green's function for (11) that decays for large $|z|$ is

$$K(z, z_0) = -(2k)^{-1}e^{-k|z-z_0|}. \qquad (13)$$

[1] The Boussinesq approximation is equivalent to taking the density scale height, H, to be infinitely large. In this limit, the density ρ_s is constant in (3) and H represents a characteristic depth for the flow in (5).

Solutions to (11) are superpositions of Green's functions for sources at $z_0 = 0$ and $z_0 = \Delta z$:

$$\phi = \frac{Qw_q}{2ik^2\Delta z}\left(\frac{e^{-k|z|}}{c} + \frac{e^{-k|z-\Delta z|}}{\Delta z - c}\right). \quad (14)$$

The dispersion relation is now obtained by requiring the solution to be consistent with the vertical velocity necessary to produce the heating. Substituting (14) in (12) and evaluating the resulting expression at $z = z_q = 0$ yields

$$(2\Delta z/Q - \mu)\,k^2c^2 + [k\Delta z(1 - 2\Delta z/Q) - \mu]\,kc + k\Delta z = 0, \quad (15)$$

where $\mu = 1 - e^{-k\Delta z}$.

It is easy to check that, for any nonzero value of the heating intensity Q, solutions to (15) are unstable in the limit of very long waves, while in the short-wave limit the waves are always stable. The effects of heating are therefore not restricted to scales smaller than a Rossby radius, as has been suggested by both Wang and Barcilon (1986) and Bannon (1986) based on the moist Eady problem.

When the heating is constrained by (10), $Q/\Delta z = 1$ and the frequency and growth rate of the wave depend only on $k\Delta z$, the ratio the depth of the heating to the decay scale of the wave. Figure 2 shows the dispersion relation in this case. The steering level of the unstable modes always lies roughly in the center of the heating ($c_r \approx \Delta z/2$) and the maximum growth rate is 0.62 at $k\Delta z \approx 1.2$, which for comparison is almost exactly twice the maximum growth rate of a dry Eady wave. At large k, a pair of neutral waves exists, one propagating somewhat faster than the flow at $z = 0$, and the other propagating slightly slower than the flow at $z = \Delta z$. Overall, the dispersion relation is distinctly similar to that of the Eady problem.

The structure of the most unstable wave is illustrated in Fig. 3, normalized so that $w_q = 1$. The streamfunction has maxima at $z = 0$ and $z = \Delta z$ and tilts westward with height for $0 < z < \Delta z$. The temperature perturbation has maxima at the same levels but tilts eastward and is discontinuous across $z = 0$ and $z = \Delta z$, reflecting the δ-function forcing in (11). Physically, there is an abrupt shift at these levels (since w is smooth) between diabatic heating partially balanced by adiabatic cooling and pure adiabatic cooling. In general, the structure for $0 < z < \Delta z$ is much like that of a developing dry baroclinic wave, although both dry baroclinic and diabatic processes directly contribute to the growth of the disturbance. The upshear tilt of ϕ implies a northward heat flux and accompanying conversion of zonal available potential energy to perturbation potential energy; at the same time, reference to Fig. 3 shows that the temperature perturbation is in phase with the heating, leading to diabatic generation of perturbation potential energy. For the most unstable mode with $Q/\Delta z = 1$, the baroclinic conversion amounts to somewhat more than 40% of the diabatic generation.

If (10) and the question of whether such combinations of Q and $G(z)$ are physically realizable are ignored, varying $Q/\Delta z$ much beyond unity significantly changes the dispersion relation and vertical structure of the waves. Dispersion relations for several different values of $Q/\Delta z$ are shown in Fig. 4. The maximum growth rate increases rapidly with $Q/\Delta z$—if $Q/\Delta z$ doubles from 1 to 2, the growth rate increases by a factor of 3.5. In addition, the phase speed of an unstable mode at a given k increases monotonically with $Q/\Delta z$ until, for $Q/\Delta z \geqslant 1.5$, the steering level lies above the region of heating (i.e., $c_r > \Delta z$). Finally, note that as $Q/\Delta z$ increases, the wavenumber of the short-wave cutoff and of the most unstable mode initially increase, so that the instability moves to smaller scales, while for $Q/\Delta z \geqslant 1.8$ both quantities decrease. In fact, it can be shown that both quantities approach zero in the limit that $Q \rightarrow \infty$, while the maximum growth rate grows without bound.

For large $Q/\Delta z$, the instability becomes essentially a CISK phenomenon and the energetics of the modes are dominated by the diabatic generation of perturbation potential energy. These changes are manifest in

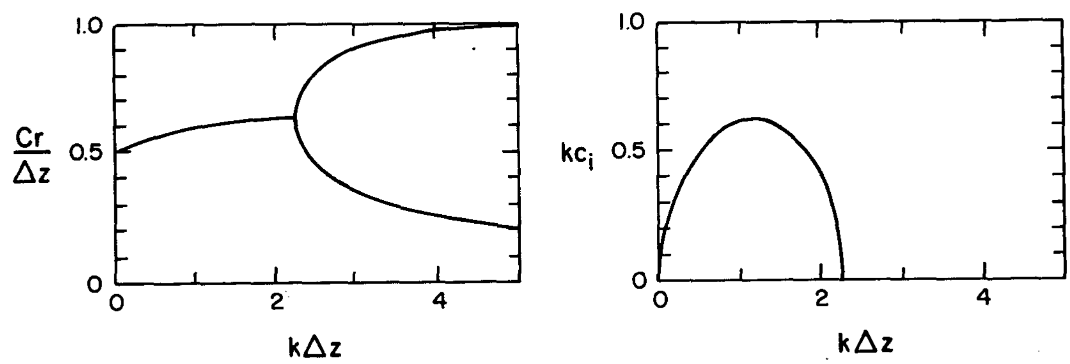

FIG. 2. The dispersion relation for the free-shear problem with heating given by (10). Note that the ordinate in each case is $k\Delta z$.

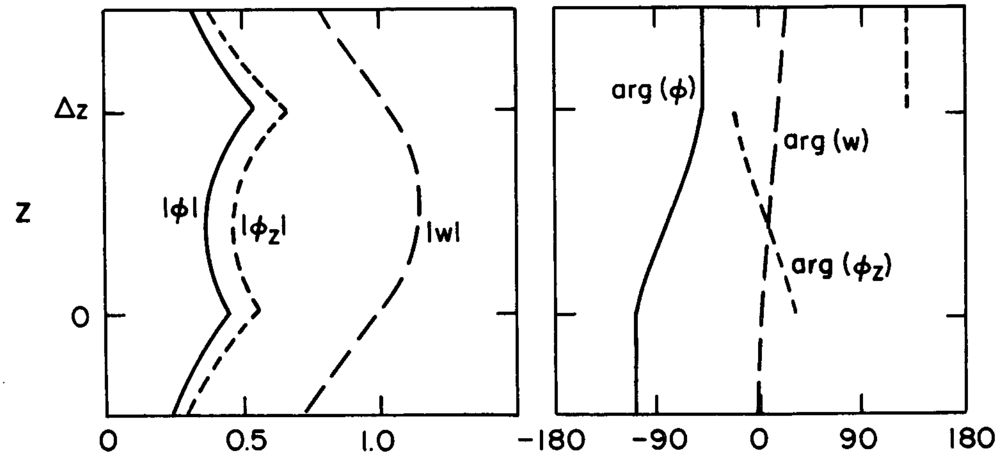

FIG. 3. The perturbation fields, normalized so that $w_q = 1$, for the most unstable mode at $k\Delta z = 1.2$, using $Q/\Delta z = 1$. When $z < 0$, both the geopotential and temperature perturbations have the same phase, but only $\arg(\phi)$ is shown.

the structure of the unstable modes (Fig. 5). Strong and highly divergent vertical motion, directly driven by the large heating, spins up cyclonic circulation at $z = 0$ and anticyclonic circulation at $z = \Delta z$ (in regions of upward motion). The baroclinic conversion is negligible because the westward tilts in ϕ occur only where $|\phi|$ is small, but diabatic generation of perturbation potential energy is large, since the wave has a warm core aligned with the heating. The characteristics of baroclinic waves that are dominated by heating are discussed in more detail in Craig and Cho (1988).

The above results for the free-shear problem reproduce the basic results of previous studies of baroclinic waves in the presence of parameterized heating. Some of these previous results show that (see Mak 1982; Moorthi and Arakawa 1985; Wang and Barcilon 1986; Bannon 1986; Craig and Cho 1988):

(i) The growth rate is sensitive to Q for a fixed heating profile. As Q increases, the most unstable mode moves to shorter waves and the maximum growth rate increases; for sufficiently large Q, the most unstable wavenumber begins to decrease with Q;

(ii) The phase speed of an unstable mode (at fixed k and for fixed heating profile) steadily increases with Q; when Q is sufficiently large, the steering level lies outside the region of nonzero heating; and

(iii) For heating that roughly satisfies (10), the unstable waves are structurally reminiscent of dry baroclinic waves and the baroclinic conversion is significant

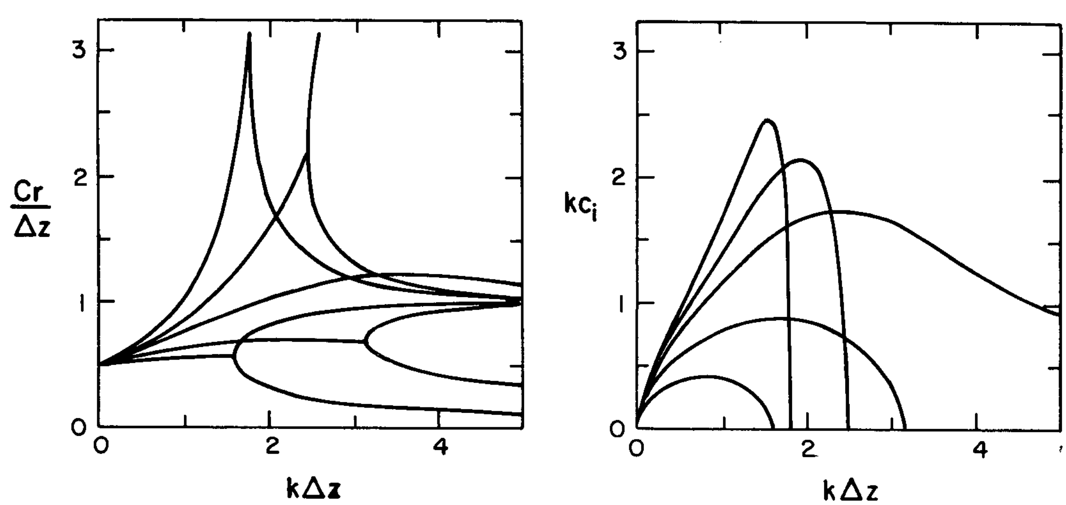

FIG. 4. Dispersion relations from the free-shear problem for $Q/\Delta z$ = 0.75, 1.25, 1.75, 2.0, and 2.25. Identification of the curves is direct as both c_r and kc_i at small k increase monotonically with Q.

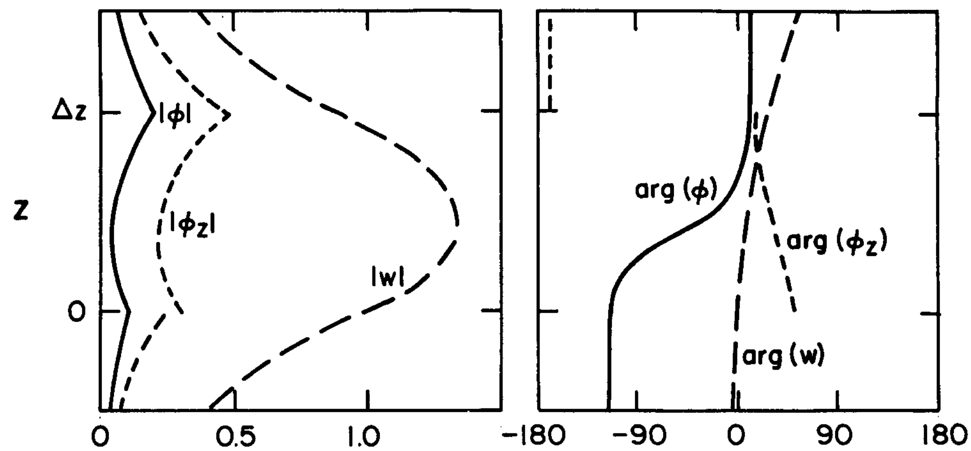

FIG. 5. The perturbation fields for the most unstable wave ($k\Delta z = 2.3$) for $Q/\Delta z = 1.8$. This value of $Q/\Delta z$ is chosen to give the maximum growth rate, $kc_i = 1.85$; and the phase speed of the mode is $c_r = 1.18\Delta z$. Again, $\arg(\phi)$ and $\arg(\phi_z)$ agree when $z < 0$.

in the modal energetics. As Q increases, diabatic effects dominate the perturbation structure and energetics.

Despite the highly idealized nature of the basic state and geometry, the unstable waves in the free-shear problem appear to be essentially similar to those in the moist Eady problem or more general settings and presumably result from a similar, if somewhat distilled, set of dynamical processes.

b. Potential vorticity dynamics

It seems clear that this simplified model captures the essential dynamics of baroclinic wave-CISK; however, what these essential dynamics are remains obscure. The unadorned potential vorticity dynamics of the free-shear problem will be used both to understand the mechanism by which the shear and heating interact to produce instability and to account for the changing properties of the instability as $Q/\Delta z$ varies.

The perturbation potential vorticity equation for the free-shear problem may be written as:

$$d(\pi')/dt = Qw_qG_z,$$

where $d/dt = \partial_t + U\partial_x$ and $\pi' = \phi_{zz} - k^2\phi$ is the perturbation potential vorticity. Because $\Pi_y = 0$ in the free-shear problem, π' changes only through diabatic generation, and is otherwise simply advected by the basic state flow.

To aid in visualizing how π' relates to modal structure, Fig. 6 shows schematic zonal cross sections of the wave fields for three values of c. Since the heating and w_q attain their maximum positive value in the center of the figure, the diabatic generation of π' is also maximized at $x = 0$, with production of positive π' at $z = 0$ and negative π' at $z = \Delta z$.

Consider the kinematics that lead to the distributions of π' in Fig. 6. In essence, positive and negative potential vorticity perturbations are advected downstream (relative to the wave) from the regions of positive and negative diabatic generation. For the neutral wave with $0 < c_r < \Delta z$ (Fig. 6a), π' has the same phase at $z = 0$ and $z = \Delta z$ (because both the advection and the diabatic source change sign from $z = 0$ to $z = \Delta z$) and is exactly in quadrature with the generation of potential vorticity. When $c_r > \Delta z$ (Fig. 6b), π' is advected to the west at both $z = 0$ and $z = \Delta z$; the resulting field shifts 180° at $z = \Delta z$ but remains in quadrature with $d\pi'/dt$. In a growing mode, π' must shift "upstream" to become partially in phase with the diabatic source term. Hence, in the case $0 < c_r < \Delta z$ and $c_i > 0$ (Fig. 6c), the lower potential vorticity pattern shifts eastward and the upper shifts westward, where the shift at either level is less than one-quarter wavelength.

Since the vertical velocity at $z = z_q$ provides the feedback between the disturbance and the generation of potential vorticity, it is crucial to understand the link between π' and w. Any distribution of perturbation potential vorticity induces a circulation, and this circulation must be consistent with the value of w_q used in (11). In order to match w_q (for π' of a given magnitude), the vertical velocity in a modal solution must be relatively large when Q is small and relatively small when Q is large.

In the free-shear problem, the induced ϕ is given by (14) and the vertical velocity may be calculated directly from (12), yielding

$$w = ik(kc + 1)\phi, \quad z \leqslant 0.$$

Thus, w_q is proportional to $|\phi(0)|$ and, as can be seen from Figs. 3 and 5, leads $\phi(0)$ by roughly a quarter wavelength, where the exact phase shift between w and ϕ depends on c_i. (More general, and more lengthy, arguments based on the quasi-geostrophic w-equation produce the same result.) It is now clear that, for π' of

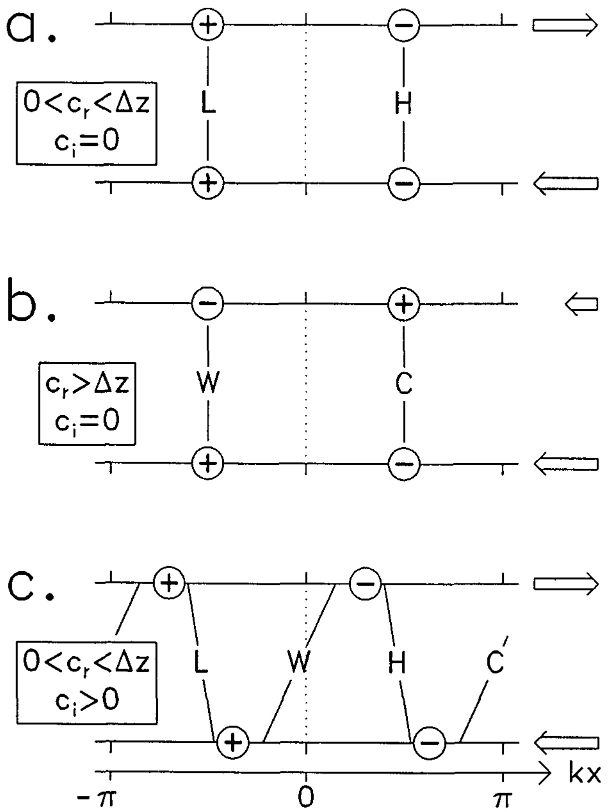

FIG. 6. Schematic x–z cross sections of wave with: (a) $0 < c_r < \Delta z$ and $c_i = 0$, (b) $c_r > \Delta z$ and $c_i = 0$, and (c) $0 < c_r < \Delta z$ and $c_i > 0$. Only the region $-\pi \leqslant kx \leqslant \pi$, $0 \leqslant z \leqslant \Delta z$ is shown in each case. The dotted line corresponds to the location of the maximum in w_q and hence, the maximum heating, while the location of the maximum positive and negative values of π' are shown by the circled plus-signs and minus-signs, respectively, and large arrows indicate the direction of zonal wind relative to the wave (i.e., the sign of $U - c_r$) at $z = 0$ and $z = \Delta z$. Phase lines of ϕ are labeled with **H** (ridges) or **L** (troughs); the phase lines of ϕ_z are labeled with **W** (warm temperatures) or **C** (cold). Phase lines for ϕ_z and ϕ are not shown in (a) and (b), respectively, since in these cases the structure is qualitatively dependent on the precise value of c_r.

a given magnitude, the arrangement of potential vorticity for a neutral mode with $0 < c_r < \Delta z$ (Fig. 6a) maximizes the induced w_q, by vertically aligning π' so that the contributions from each level are in phase. Conversely, the distribution of π' shown in Fig. 6b, which is consistent with a neutral mode having $c_r > \Delta z$, minimizes $|w_q|$, since the circulation induced by $\pi'(\Delta z)$ tends to cancel that induced by $\pi'(0)$. The distribution of potential vorticity shown in Fig. 6c represents an intermediate situation: the phase shift in the pattern of π' (introduced by nonzero c_i) leads to a reduction in the reinforcement of the induced circulation, and $|\phi(0)|$ and $|w_q|$ are smaller than in the case with $0 < c_r < \Delta z$ and $c_i = 0$. Similar arguments show that an unstable mode with $c_r > \Delta z$ will produce slightly larger vertical velocity at $z = 0$ than in the case that the wave is neutral and has $c_r > \Delta z$.

Some of the characteristics of the unstable modes can now be diagnosed. When the heating is not too large (in particular, for consistent heating with $Q = \Delta z$), waves necessarily have $0 < c_r < \Delta z$, so that the mode may produce sufficient vertical velocity to match the heating. At large wavelengths, the circulations induced by the potential vorticity at $z = 0$ and $z = \Delta z$ are deep, and the reinforcement of $\phi(0)$ and w_q is strong enough that the wave must tilt with height and have nonzero c_i. For larger k, the interaction between the potential vorticity at different levels wanes, resulting (much as in the dry Eady problem) in a short-wave cutoff followed by neutral "edge" waves confined near $z = 0$ or $z = \Delta z$. As Q increases with Δz fixed, the response for any given c also increases, and the modes must then adjust their structure to be less efficient in forcing w (to maintain consistency with the heating). The adjustment occurs through the steady increase of c_r, and also through modification of c_i, resulting in a gradual change from configurations of perturbation potential vorticity that favor vertical motion ($0 < c_r < \Delta z$) to those that inhibit vertical motion ($c_r > \Delta z$). The dependence of the energetics of the unstable modes on the heating intensity is directly attributable to the alterations in the structure of the mode as Q is varied and again relies on the essentially different configurations of π' when $0 < c_r < \Delta z$ or $c_r > \Delta z$.

The foregoing analysis indicates that the presence of shear has a profound impact on how the heating generates vertical motion, allowing various configurations of π'. Without shear, there exists only one possible arrangement of π', and the vertical velocity is independent of c. If U is constant, the vertical velocity at z_q is

$$w(0) = \frac{Qw_q}{2\Delta z}(1 - e^{-k\Delta z}).$$

One can see that, when vertical shear is absent, only heating of magnitude $Q/\Delta z \geqslant 2/\mu$ can produce sufficient response to maintain itself, where μ is defined following (15).[2] Thermodynamically consistent heating with $Q = \Delta z$, in particular, is unable to maintain a pure CISK feedback, forcing vertical motion of only half the necessary amplitude. Clearly, the instability in this model requires the cooperative dynamic influences of both baroclinic shear and heating.

One may also readily interpret the free-shear problem within the framework developed by Hoskins et al. (1985) for the propagation and instability of ordinary Rossby waves. The key is to note that, in the free-shear

[2] Craig and Cho (1988) have termed this value $Q(k)$ the "CISK threshold." However, it can be shown from (15) that, when $Q/\Delta z$ is large, wavenumber k is stable when $Q > Q(k)$. Thus, it appears that when shear is present, $Q(k) = 2\Delta z\mu^{-1}$ instead represents a threshold beyond which the heating is *too large* and there are no configurations of π' that have sufficient cancellation to allow a consistent unstable mode.

problem, the induced vertical velocity at $z = z_q$ determines the production of perturbation potential vorticity throughout the fluid, in contrast to the usual case in which π' evolves according to the advection of Π_y by the induced horizontal velocity. Because π' and w_q tend to be phase shifted by a quarter wavelength (as discussed previously), the diabatic generation of potential vorticity acts principally to propagate the pattern of π'—the neutral waves in Fig. 2 are a manifestation of this effect and correspond to the neutral Eady waves discussed in Hoskins et al. (The direction of propagation in our case is determined by the sign of $\partial G/\partial z$, rather than by the sign of Π_y.) Given these points, the instability mechanism described in Hoskins et al. (p. 922) can be immediately applied to the free-shear problem, as long as "induced velocity" is understood to mean $w(z_q)$ rather than $v(z)$.

4. Discussion

We have analyzed the minimal problem in which baroclinic wave-CISK is possible and proposed a thermodynamic constraint on the intensity and vertical extent of the parameterized heating. This constraint serves to identify the most likely regime for the resulting dynamics.

Perhaps the most basic result is that parameterized heating can act as a dynamical surrogate for basic state potential vorticity gradients and produce a "moist" baroclinic instability, even in situations that are stable in the absence of heating. When heating is present, a disturbance may create and intensify distributions of perturbation potential vorticity without the necessity of basic state potential vorticity gradients. If, in addition, there is also vertical shear in the basic state, the differential advection permits the potential vorticity perturbations at $z = 0$, Δz to become phase locked and form an unstable mode. While this view of the dynamics is akin to that of Hoskins et al. (1985) for dry baroclinic instability, an alternative interpretation in terms of wave overreflection (see Lindzen 1988) also seems possible. According to this interpretation, the boundaries in the Eady problem constitute thin bounded wave propagation regions that permit reflection of wave energy; the potential vorticity generation associated with CISK-type heating provides an obvious surrogate for reflection. The fact that the unstable waves in the free-shear problem do not depend on a basic state potential vorticity gradient suggests that, in more general settings with boundaries and nonzero Π_y, moist baroclinic instability is not necessarily the result of a cooperative interaction between the parameterized heating and dry baroclinic instability, but rather that the presence of heating introduces distinct effects that may compete with dry baroclinic instability.

As in previous studies of wave-CISK, the growth rate and structure of the unstable waves in the free-shear problem are found to be sensitive to the magnitude and vertical profile of the heating. However, the thermodynamic constraint (10), relating the depth of the parameterized heating to the basic state stratification and specific humidity, determines the appropriate parameter range, which in the case of the free-shear problem is $Q \approx \Delta z$. In this regime, the dispersion relation and structure of the waves are reminiscent of dry Eady waves (with boundaries at $z = 0$ and $z = \Delta z$), although the heating provides an additional energy source and the maximum growth rate is increased substantially. Nevertheless, the unstable waves still result from the interaction of π' and the heating through the induced vertical velocity (rather than from advection of the basic state potential vorticity as in the dry Eady problem) and are not simply baroclinic waves with increased growth rates due to a reduction in the effective static stability by the heating.[3]

As Q is increased to values much larger than Δz (say $Q \geqslant 1.5\ \Delta z$), both the growth rate and phase speed of the most unstable mode increase and the structure of the unstable waves adjusts to produce less vertical motion, accommodating the greater feedback between the heating and vertical motion. Only in this parameter range does the modal structure approach that of a pure CISK disturbance with negligible baroclinic conversions of zonal available potential energy. Indeed, based on (10), we question whether this parameter range is physically realizable and emphasize that the unstable waves will reflect both baroclinic and diabatic processes in the most likely parameter regime, $Q = \Delta z$. Interestingly, observed disturbances in which moist convection and latent heat release appear to be important (such as rapidly deepening synoptic-scale cyclones and polar lows) generally exhibit both baroclinic and diabatic characteristics.

Of course, the estimate of the depth of the convection, which provides the basis for the specification of the heating, is based on a very simple, one-dimensional cloud model and is not beyond reproach. However, even if (10) is ignored, the free-shear problem still reveals the essential dynamics of the unstable modes. Although highly simplified, the free-shear problem illustrates how a wave can interact with both vertical shear and parameterized heating and suggests that the most important parameters governing this interaction are the magnitude of the heating and the vertical scale of the potential vorticity generation produced by the heating.

As an additional remark, we note that the scale selection in the free-shear problem depends strongly on both Q and Δz. Because Δz is the only height scale in the problem, horizontal lengths scale with $N_0 \Delta z / f$, and

[3] If the heating at any z is proportional to $w(z)$, instead of w_q, then it may in fact be formally replaced by a reduced static stability; Eady (1949) modeled the latent heat release in a layer of cloudy air in this manner. Even in this case, however, the resulting unstable waves are dependent on the potential vorticity gradient introduced by the variations in the effective N^2 and are not due, per se, to the reduction in static stability in the layer.

therefore, (at fixed $Q/\Delta z$) the wavenumber of the most unstable mode, $k_{\max}$, is such that $k_{\max}\Delta z$ is constant. It is generally held that when strong latent heat release or moist convection is present, observed cyclones develop on smaller scales than expected from dry baroclinic considerations. When $1.5 \leqslant Q/\Delta z \leqslant 2.0$, the free-shear problem does predict that the fastest growing modes will be at short synoptic or subsynoptic scales when Δz is comparable to or smaller than the depth of the troposphere, through this parameter range for Q is extreme. However, if $Q \approx \Delta z$, reference to Figs. 2 and 4 shows that $k_{\max}\Delta z$ is close to 1. Thus, in this regime, the most unstable wave has a scale of roughly $N_0\Delta z/f$ and only when the convection is shallow compared to the depth of the troposphere (Δz significantly less than H) will the moist waves be significantly shorter than typical unstable, dry baroclinic waves.

The presence of horizontal boundaries, nonzero potential vorticity gradients in the basic state, or continuous heating profile will certainly have strong quantitative impact on the unstable waves obtained here. Horizontal boundaries will directly reduce the feedback between the wave and the heating by inhibiting vertical motion and will thus moderate the effects of the heating on the structure and energetics of the wave, though this effect will be most important for the longest waves (which decay only slowly with height). As previously noted, basic states with $\Pi_y \neq 0$ and some form of parameterized heating possess two mechanisms for instability; one based on the advection of the basic state potential vorticity as in the dry Eady problem and one based on the diabatic generation of potential vorticity as in the free-shear problem. These mechanisms will interact, either reinforcing or canceling one another, and this interaction may result in either stabilization or destabilization of unstable waves. Use of a continuous $G(z)$ does not change the nature of the arguments presented here, except that the distribution of perturbation potential vorticity will then be most strongly influenced by the structure of $\partial G/\partial' z$ near the steering level. Results for the general case with both smooth heating and nonzero Π_y will be presented in a later paper.

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APPENDIX

Further Discussion of the Thermodynamic Constraint

At least three possible factors may significantly affect the constraint on Δz given by (10). They are (i) that moisture convergence may occur above the mixed layer (i.e., for $z > z_q$, where $q = 0$ was assumed), (ii) that the wave-induced variations in surface fluxes may figure importantly in the perturbation heat and moisture budgets, and (iii) that the neglected effects of entrainment of environmental air into and detrainment of moist air from the clouds may influence the depth of the clouds. Each is discussed in turn:

(i) If, in addition to that in the mixed layer, moisture convergence also occurs above the mixed layer, the vertically integrated heating will be increased without affecting our estimates for the depth of the clouds. If the convergence above $z = z_q$ is approximately in phase with w_q, then such convergence may be modeled by simply increasing q_0 by an appropriate factor. If one supposes that $z_q \approx 1$ km, a fair upper bound on the "extra" moisture convergence is arguably about 50% of that occurring below z_q, due to the rapid decrease of both q and wave convergence with height, as well as the fact that potentially buoyant air is usually confined to levels well below 2 km. More generous estimates, suggesting increases of as much as 110%, have been made by Craig and Cho (1988) based on observed polar low cases. While it is clear that no definitive argument has been made, our assertion is that values of Q that are more than 50% larger than that given by (8) (where q_0 is the specific humidity in the mixed layer) at best represent extreme cases and may have no physical significance.

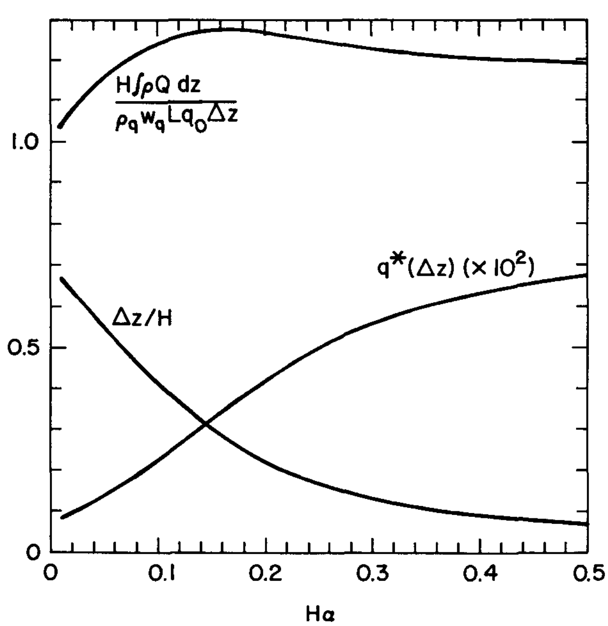

FIG. A1. The nondimensional depth of the clouds, $\Delta z/H$, the saturation specific humidity at cloud top, $q^*(z_q + \Delta z)$, and the ratio of the nondimensional, vertically integrated heating to the nondimensional depth of the clouds, $H\int \rho\dot{Q}dz/(\rho_q w_q L q_0 \Delta z)$, as functions the nondimensional entrainment parameter, $H\alpha$. The calculation assumes that mixed layer air is at 15°C and is saturated and that the lapse rate above the mixed layer is 5.5°C km^{-1}.

(ii) The neglect of surface fluxes is not easy to justify over the extratropical oceans, especially in the case of cold-air outbreaks over warm currents where the boundary layer temperature and specific humidity may be far from the SST and the associated saturation specific humidity. One may estimate the surface evaporation from the bulk aerodynamic formula

$$E = \rho_0 c_E |\mathbf{v}_0| (q_s^* - q),$$

where c_E is the transfer coefficient and q_s^* is the saturation specific humidity at the sea surface temperature. Taking $c_E = 10^{-3}$, $|\mathbf{v}_0| = 5$ m s^{-1}, and $q_s^* - q = 4$ g kg^{-1} results in $E = 2$ mm d^{-1} m^{-2} or a latent heat flux of 50 W m^{-2}; the difference in specific humidities is generous, corresponding to half the saturation value at 10°C. A similar calculation gives a sensible heat flux of 25 W m^{-2}, assuming that the air–sea temperature difference is 5°C. (Note that such estimates are for the fluxes due to a small amplitude linear wave, and are not typical of the mature phase of intense maritime cyclones.) These values may be compared with the precipitation rate given by (7); using 80% of the saturation specific humidity at 10°C for q_0 ($q_0 \approx 6$ g kg^{-1}) gives $P = 9$ mm d^{-1} m^{-2}, where $\bar{w}'_q$ is scaled directly from (5) with $u_0 = 5$ m s^{-1} and $m = 3 \times 10^{-3}$ s^{-1}. Thus, the surface fluxes are smaller than the moisture convergence by the wave or the parameterized heating, though this estimate is certainly sensitive to both the magnitude of $q_s^* - q$ and of the transfer coefficients. Even in cases of larger surface fluxes, it seems clear that the fluxes will not be in phase with the moisture convergence and hence, that their contribution to the perturbation moisture budget cannot be modeled as an effective increase in the moisture convergence as in (i).

(iii) A rough estimate of entrainment effects can be made based on a simple, steady 1-D cloud model, assuming an initial cloud mass flux at the top of the mixed layer, entrainment of dry air into the cloud at a constant fractional rate α (i.e., $\alpha = M_c^{-1}\partial M_c/\partial z$, with M_c the cloud mass flux), and detrainment of all cloud mass flux when the temperature of cloud air is equal to that of the environment. For a given temperature profile in the environment, one can then readily calculate the depth of the resulting cloud based solely on budgets of heat and moisture for the cloud. Figure A1 shows the depth of the clouds, the amount of water vapor detrained at cloud top (assuming cloud air saturated at the environmental temperature), and the ratio of the nondimensional, vertically integrated heating and nondimensional cloud depth all as functions of the entrainment parameter α. Although both the depth of the clouds and the amount of q detrained from the clouds are substantially changed by entrainment, especially for $\alpha \gtrsim 0.2$, the ratio of the integrated heating to the depth of the clouds remains nearly equal to the value given by (10) for nonentraining clouds. Thus, entrainment effects may alter the dependence of Δz on q_0 but, for a given value of Δz, produce little change in the value of Q implied by (10).

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