Tropical Wave-CISK with a Moisture Budget and Cumulus Friction

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ABSTRACT

The wave-CISK problem is investigated in a linearized context with two significant departures from previous models: 1) the cumulus heating parameterization is made thermodynamically consistent by requiring that cumulus heating exactly equal the latent heat of moisture converged in the wave; and 2) cumulus friction is included in the model. The parameterization of vertical transport of horizontal momentum by cumulus clouds follows the formulation of Schneider and Lindzen (1976); with a constant mean zonal wind, cumulus friction is proportional to the mean cloud mass flux.

Results from the model of Stevens et al. (1977) are presented for various scales of motion. For magnitudes of cloud mass flux typical of the ITCZ, no wave-CISK modes appear on the synoptic scale of several thousand kilometers. As the horizontal scales of these long-period waves are decreased, the disturbances become frictionally controlled, neutral wave-CISK modes. When the mean cloud mass flux is reduced by a factor of 2, instabilities do occur at synoptic scales.

Kelvin waves are simulated on a midlatitude f-plane with a zonal wall. For very long space and time scales, they are frictionally dominated and neutral modes. At very short time scales, they act as gravity waves.

Gravity waves with the scales of observed squall lines are largely unaffected by cumulus friction. They display the instability characteristics of previous inviscid studies. By contrast, very slow gravity waves are neutralized by the action of cumulus friction.

1. Introduction

This paper treats the wave-CISK (Conditional Instability of the Second Kind) problem, i.e., finding those tropical waves that can be self-maintained by the latent heat released in cumulus convection, which is organized on the wave scale by the wave fields. As in previous studies (Yamasaki, 1969; Hayashi, 1970; Lindzen, 1974, Kuo, 1975; Chang, 1976; Stark, 1976), linearization is assumed for a simplified wave analysis. Also, the mean zonal wind is assumed to be constant in space and time, so that the intrinsic time scale for the wave is the Doppler-shifted period.

The model used here differs in two important respects from earlier treatments:

1) The parameterization of cumulus convection takes account explicitly of a moisture budget. The CISK condition on vertical velocity at the top of the moist layer follows directly from relating energy release in condensation to the moisture source for the wave.

2) Cumulus friction is included in the model. Stevens et al. (1977; hereafter referred to as SLS) found that cumulus friction plays a crucial role in the dynamics of synoptic-scale disturbances. In particular, cumulus friction is necessary to obtain the extremely small ratios of temperature amplitude to heating typical of observed tropical waves. In addition, it greatly reduces sensitivity to the shape of cumulus heating. It will be shown here that vertical mixing of momentum by cumulus convection also significantly alters the wave-CISK solutions.

The mean cumulus activity appears explicitly in the perturbation equations only in the cumulus friction terms. A basic state with mean cumulus activity implies that negative perturbation heating by cumulus convection has a well-defined physical meaning: although negative in the perturbation sense, the total cumulus activity is still positive so long as the amplitude of the perturbation is less than the mean amplitude. The precipitation field in the composited wave of Reed and Recker (1971), shown in their Fig. 12, supports this view. Thus the dilemma faced by previous investigators of wanting to use so-called "conditional heating" (without a negative component), while the simple linearized models admit only unconditional heating, is avoided with a nonzero basic state.

Surface friction and turbulent mixing in the subcloud layer are characteristic processes in the observed tropical atmosphere. They are parameterized in the present model, as described in SLS. Thus the term wave-CISK is used in a generalized sense: moisture
convergence results from both the inviscid wave field and surface drag (i.e., Ekman-like pumping).

In the following section, a simple treatment of the vertically integrated heat and moisture budgets leads to a relationship between total heating and vertical velocity at the top of the moist layer, which provides a consistency condition for the wave-CISK eigenvalue problem. Section 3 compares the traditional eigenvalue method with the forcing calculation used here and discusses the implications for unstable growth, equilibration and storage of moisture in the wave solutions. In Section 4 we specify the functional form of the heating, consistent with both a non-entraining cumulus model and a more general cloud model of Appendix A. Section 5 reviews the major results of previous inviscid investigations in the context of our forcing calculation.

Sections 6–9 present the results of the present wave-CISK study. Section 6 treats the low-frequency waves observed in the tropics (as weather) with cumulus momentum mixing of magnitude typical of the Marshall Islands region, considered by Reed and Recker (1971). The results suggest that significant instability may occur when there is less mean cumulus activity; such a calculation is presented in Section 7. The final two sections discuss the influence of cumulus friction on other types of waves observed in the tropics: low-frequency Kelvin waves (Section 8) and high-frequency gravity waves (Section 9). Many of the results can be understood in the context of a highly idealized Boussinesq model, in which the perturbations can be mathematically described by a set of ordinary differential equations with constant coefficients; this simple model is discussed in Appendix B.

2. Parameterization of convective heating and the instability condition

The premise of wave-CISK is that moisture convergence associated with the wave fields provides the water vapor which condenses in cumulus clouds, thereby heating the atmosphere on the time and space scales of the wave; the heating concurrently forces a wave response with the moisture convergence needed to maintain the heating. In this study we explicitly relate the moisture budget and heat budget to derive a consistency condition between moisture convergence and heating.

Because of the assumptions to be made shortly on the moisture structure, the consistency condition will involve a vertical velocity \( w \). In contrast with earlier studies, this condition has nothing to do with a Lagrangian notion of moist air parcels being lifted to their lifting condensation level; rather, it results strictly from a budget statement. The Lagrangian approach seems inconsistent with the existence of turbulent mixing in the moist layer which is always bringing moist parcels to their lifting condensation level. The convergence organizes the deep cumulus convection by supplying moisture, not by large-scale uplifting.

The time-averaged thermodynamic structure of the atmosphere is approximated by a series of horizontally homogeneous layers. The mean temperature profile is that specified in Section 5 of SLS and depicted in Fig. 4 of SLS: a subcloud mixed layer with vanishing static stability, an isothermal stratosphere, and the intermediate cloud layer with a constant [in \( s^* = \log(\rho_0/\rho) \) coordinates] static stability. Consistent with a surface temperature of 27°C, moist surface air with specific humidity 17 g kg\(^{-1}\) and an interior static stability \( \Gamma = (\overline{T}/\overline{\theta})(d\overline{\theta}/ds^*) = 20.2 \) K, we take cloud base to be the lifting condensation level \( s_c^* = 0.07 \) (altitude 600 m) and assume the tropopause to be the top of the heating region \( s_T^* = 2.01 \) (altitude 14.5 km), where surface parcels lose their buoyancy in moist adiabatic ascent.

Observations of the moisture structure by Malkus (1958) and Augstein et al. (1974) show that the specific humidity is fairly well mixed up to the level of trade inversion, above which it decays very rapidly with height to a small value. We specify a constant specific humidity \( (\dot{q} = 12 \) g kg\(^{-1}\)) for the moist layer, which extends in our model to an altitude of 2 km \( (s_c^* = 0.25) \). The decrease in specific humidity at the top of the subcloud mixed layer is neglected in this formulation; \( \dot{q} \) may be considered to be a vertically averaged moisture parameter representative of the entire moist layer. A more general approach is described in Appendix A.

The depth of the moist layer is chosen to be consistent with the observation by Reed and Recker (1971) that significant moisture convergence occurs in the wave trough up to an altitude of about 2 km. This choice differs from previous wave-CISK studies (Yamasaki, 1969; Hayashi, 1970; Lindzen, 1974; Kuo, 1975; Chang, 1976), which have considered moisture convergence to be restricted to the subcloud layer.

Our statement of a moisture budget for the wave is guided by the finding of Cho and Ogura (1974) that the deep cloud activity, which is responsible for essentially all the precipitation, is highly correlated with the moisture convergence. In fact, there is approximate equality between vertical moisture flux through deep clouds and low-level moisture convergence in Cho and Ogura’s observational results [their Eq. (3.1)], provided one takes the moist layer to be two to three times deeper than the subcloud layer and assumes that the large-scale vertical mass flux is proportional to height in the lower troposphere—in agreement with Reed and Recker’s observations. Here we assume that all the water vapor converged (horizontally) by the wave fields enters the deep clouds, condenses and precipitates. No storage occurs, either through a change in the specific humidity \( (\partial \theta / \partial t = 0) \) or by a deepening of the moist layer \( (\partial s^* / \partial t = 0) \). For simplicity, surface evaporation is assumed to be unaffected by wave activity. The time-dependent part of the large-scale moisture budget
for a column with unit horizontal area is then

$$P = -\int_0^\infty \nabla \cdot (\bar{p} \tilde{v}) ds,$$  \hspace{1cm} (2.1)

where $P$ and $\tilde{v}$ are the perturbation precipitation and horizontal velocity fields and $\bar{p}$ is the mean density field, assumed to be a function of height only. Since the vertical velocity $\langle w \rangle$ must vanish at the surface, the assumed moisture structure and the moisture budget can be combined for a simple relationship between precipitation and vertical velocity $w_q$ at the assumed top of the moist layer

$$P = -\tilde{q} \int_0^{z_q} \nabla \cdot (\bar{p} \tilde{v}) ds = -\tilde{q} \int_0^{z_q} \frac{\partial}{\partial z} (\bar{p} \langle w \rangle) ds = \tilde{q} \bar{p},$$  \hspace{1cm} (2.2)

The anelastic approximation to the continuity equation

$$\nabla \cdot (\bar{p} \tilde{v}) + \frac{\partial}{\partial z} (\bar{p} \langle w \rangle) = 0$$  \hspace{1cm} (2.3)

has been applied.

In our statement of the perturbation heat budget for a column of unit area, we neglect radiational cooling and surface heat flux on the wave scale. Then the time-dependent apparent large-scale heating ($Q_1$), integrated through an atmospheric column, is derived from the net condensation minus evaporation, which equals the precipitation at the surface:

$$\int_0^\infty c_p Q_1 ds = L_0 P.$$  \hspace{1cm} (2.4)

Here $c_p$ is the heat capacity at constant pressure and $L_0$ the latent heat of vaporization or condensation.

For a given magnitude and phase of the moisture convergence, Eqs. (2.2) and (2.4) determine the amplitude and phase of the vertically integrated heating. The shape of the heating function is yet to be determined. As in all previous treatments except the recent study by Stark (1976), we prescribe the vertical dependence of the time-dependent heating in all calculations reported here. No feedback is allowed by which the wave fields can affect the functional form of the heating.

With the specification of the heating shape, there is sufficient information to define the instability calculation. The consistency condition is that the wave solution has a convergence field which is consistent with the heating field that drives it; i.e., the calculated vertical velocity $w(z_q^*)$ at the top of the moist layer that is forced by the heating must equal the vertical velocity $w_q$ that is explicitly in the heating function, i.e.,

$$w(z_q^*) = w_q,$$  \hspace{1cm} (2.5)

both in amplitude and phase. The temporal and horizontal spatial structure is presumed to be already separated out by appropriate Fourier or normal mode decomposition.

3. Eigenvalue versus forcing problem

As an instability problem wave-CISK has been solved by finding those eigenvalues which meet a consistency condition. The eigenvalue is the complex (Doppler-shifted) frequency $\omega$, of which the imaginary part gives the growth rate. Since we are considering linear dynamics the absolute magnitude of the convergence does not enter the instability analysis.

In reality, exponential growth of the wave disturbance cannot occur indefinitely because of the finite amount of moisture available for condensation heating. Our basic state (without any disturbances) consists of mean cumulus activity with cloud mass flux $\bar{M}_c$ and associated precipitation $\bar{P}$ which is statistically steady in space and time. With our parameterization assumption of no moisture storage, the fact that negative clouds do not exist limits wave growth to a finite amplitude no greater than the magnitude of mean cumulus activity and precipitation. The scope of this research does not include details of the equilibration process, namely, the dynamical mechanism whereby the wave ceases to grow beyond this finite amplitude; but we will consider shortly important implications for the moisture budget.

An alternative procedure to the eigenvalue method is to treat the same system as a forcing problem, in which the real frequency of the heating is specified, and then consider the response in the field that gives the consistency condition. We will note some advantages to this approach. Thus, the search for complex $\omega$ which satisfy the condition

$$\text{response} = \text{forcing}$$

or

$$w(z_q^*) = w_q$$  \hspace{1cm} (3.1)

is replaced by a more limited search over real $\omega$ which satisfy the condition

$$\text{phase of response} = \text{phase of forcing}$$

or

$$\text{phase } [w(z_q^*)] = \text{phase } [w_q].$$  \hspace{1cm} (3.2)

We will see that, when (3.2) is satisfied, the condition $|w(z_q^*)| > |w_q|$ or $|R'| \equiv |w(z_q^*)/w_q| > 1$ implies instability, while $|R'| < 1$ implies stability. Moreover, large values of $|R'|$ are generally associated with greater growth rates in the more conventional instability analysis. While the relation between growth rate (in the conventional approach) and $|R'|$ (in the problem with real $\omega$ and phase match) is not precise, the latter approach is vastly simpler; it also serves to clearly delineate unstable and stable modes. In addition, observed waves tend to be neither growing nor decaying with time; for such waves $|R'|$ may be viewed as a measure of the efficiency of CISK maintenance. The
relationship between the two approaches to instability is illustrated most simply by a calculation similar to that in Lindzen (1974).

We consider a linearized, two-dimensional, hydrostatic gravity wave in an incompressible Boussinesq fluid on a nonrotating plane, i.e.,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{1}{\rho_0} \frac{\partial p}{\partial x} \\
\frac{\partial p}{\partial z} &= -\rho g \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial p}{\partial x} + \frac{\partial \rho_0}{\partial z} &= -\frac{\partial \rho_0}{\partial t} \\
\frac{\partial \rho_0}{\partial z} &= -Q
\end{align*}
\]  

(3.3)

where \(u\) and \(w\) are perturbation horizontal and vertical velocities, \(\rho\) is density perturbation, \(g\) the gravitational acceleration, \(t\) time, \(x\) the horizontal coordinate, and \(z\) the vertical coordinate. \(Q\) is an equivalent mass sink (cf. Lindzen, 1974), analogous to heating in the compressible atmosphere. As in the stratified, incompressible Boussinesq approximation, basic-state density \(\rho_0\) and stratification \(\frac{d\rho_0}{dz}\) are both taken to be constant in defining the Brunt–Väisälä frequency

\[N^2 = -\left(\frac{g}{\rho_0}\right) \frac{d\rho_0}{dz}.\]

We assume there is no mean flow. Suppose the heating is given by

\[Q = \begin{cases} \alpha^{-1} w_0 (d\rho_0/dz) \exp[i(kx + \omega t)], & z_e < z < z_T \\ 0, & \text{elsewhere} \end{cases}, \]

(3.4)

where \(k\) is horizontal wavenumber, \(\omega\) the frequency, and \(\alpha\) a parameter measuring the degree of feedback between moisture convergence and condensation “heating” \(Q\). This very simple functional form (\(Q\) constant with height in the cloud layer) corresponds to the single non-entraining deep cumulonimbus parameterization of Lindzen (1974). With a moisture budget, \(\alpha = 1\) represents a complete conversion of converged moisture into precipitation, as in the previous section. For \(\alpha > 1\), proportionally less \((\alpha^{-1}/\alpha)\) of the water vapor feeds back into the heating. For \(\alpha < 1\), more condensation heating is realized than there is moisture available—clearly a physically unviuable situation.

The vertical structure equation for this system is

\[\frac{d^2 w}{dz^2} + \lambda^2 w = \lambda^2 (\alpha^{-1} w_0), \]

(3.5)

where the vertical wavenumber \(\lambda\) and Doppler-shifted frequency \(\omega\) are related by \(\lambda^2 \omega^2/k^2 = N^2 > 0\). Since we are considering a motionless basic state, \(\omega = \omega_0\). With boundary conditions

\[w(0) = 0 \quad \text{and} \quad w(z) \sim e^{i\lambda z} \quad \text{as} \quad z \to \infty \quad (\lambda_r \omega_r > 0) \quad (3.6)\]

\[w(z)\text{ in the cloud interior } z_e < z < z_T\text{ is}

\[w(z) = \alpha^{-1} w_0 \left[ e^{i\lambda z} (\cos \lambda z - \cos \lambda z_e) + i \sin \lambda z (e^{i\lambda z} - e^{i\lambda z_e}) \right], \]

(3.7)

where \(w(z_T) = w_0\) is the eigenvalue consistency condition.

For a given \(\alpha\), there are an infinite number of discrete eigenvalues \(\lambda_n(\alpha)\), \(n = 0, 1, 2, \ldots\). If the parameter \(\alpha\) is allowed to vary smoothly, then the analytic functions \(\lambda_n(\alpha)\) also vary continuously. Fig. 1 displays the eigenvalue solutions \(\lambda_n(\alpha)\) for \(n = 0, 1, 2, 3, 4, 5\) in the complex \(\lambda\) plane. As in Section 2, \((\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) = (0.07, 0.25, 2.01)\) are assumed, and \(z = \bar{h} x^*\) with an average
Fig. 2. Amplitude and phase of the response $R'$, corresponding to the real $(\lambda_n)$ axis of Fig. 1. The upper scale of amplitude denotes the magnitude $|R'|$; the lower scale indicates the percentage excess ($A'$) of the response amplitude $|w(z_q)|$ over the forcing $w_q$. Circles correspond to the real $\lambda_n$ wave-CISK solutions of Fig. 1.

tropospheric scale height $H=7.5$ km relates the Boussinesq and log-$p$ coordinates. $\lambda_r$ and $\lambda_i$ indicate the real and imaginary parts of $\lambda$, respectively. The value of $\alpha$ is denoted at various points along the $\lambda_n(\alpha)$ solution curves.

In this simple model, growth rate ($-\Delta\lambda$) is related to the complex vertical wavenumber ($\lambda$) by

$$-\Delta\lambda = \frac{\lambda_i}{\lambda_r^2 + \lambda_i^2} |k| N. \quad (3.8)$$

Thus $\lambda_i[\leq, =, \geq] 0$ indicates a [stable, neutral, unstable] solution for a specified $\alpha$. The loci of constant growth rate in the complex $\lambda$ plane are circles with center on the $\lambda_i$ axis and circumference passing through the origin; the smaller the radius of such a circle, the larger the growth rate. With $\alpha=1$, only the $n=2$ solution displays a finite growth rate. The $n=5$ solution is most unstable for $1^\circ < \alpha \lesssim 1.6$. The $n=3$ solution is most unstable for $1.6 \lesssim \alpha \lesssim 2$. All solutions are stable for $\alpha > 2$, i.e., when less than half the moisture convergence goes into heating.

The neutral modes ($\lambda_i = \Delta\lambda = 0$) have a second interpretation. Consider the vertical structure equation to be the expression of a forced problem in which $w_q$ is the specified magnitude of the forcing (heating) and $\alpha$ is provisionally set to unity from a moisture budget with no storage; $w(z_q)$ is the calculated response to the specified heating. Fig. 2 shows the amplitude and phase of the ratio $R'$ of the response $w(z_q)$ to the forcing $w_q$ as $\lambda$ is varied along the real axis of Fig. 1; due to the linearity of the problem, the absolute magnitude of $w_q$ does not enter. Those $\lambda$ for which the phase of the response is identical to the phase of the forcing (namely, zero) are the same neutral solutions of the eigenvalue problem. When the phase of the response (moisture convergence) is locked to the phase of the forcing (heating), the ratio

$$R' = \frac{w(z_q)}{w_q} \quad (3.9)$$

of the forcing calculation is related to the feedback parameter $\alpha$ of the eigenvalue neutral solutions by $R' = \alpha$. With phase lock, $R'$ can be interpreted as the ratio of the amount of moisture converged by the wave to the amount of moisture that provides the heating to drive the wave. $R'<1$ clearly implies that moisture convergence is insufficient to maintain the disturbance.

For a given $\alpha$, each unstable solution can be related to a forced solution with larger $R'=\alpha$ and the same index $n$ [i.e., along a $\lambda_n(\alpha)$ curve of Fig. 1]; hence, all the unstable solutions can be identified by first locating the forced solutions for which phase ($R'=0$). The converse is not true; finding all the unstable solutions for a given $\alpha$ (e.g., $\alpha = 1$), which has been the traditional wave-CISK procedure, does not necessarily yield a complete set of unstable and/or neutral solutions for other $\alpha$. Specifically, the $n=3, 4, 5$ modes can be self-maintained (either neutral or unstable) with precipitation less than moisture convergence ($\alpha > 1$) even though an $\alpha = 1$ solution is nonexistent, as in the $n=4$ case, or stable as for $n=3, 5$. Although the search over real wavenumber or frequency for phase-locked solutions is more limited in scope, it can provide more general results.

The growth rate is one piece of information not available with the forcing calculation. However, the nature of the observed waves makes this parameter more or less irrelevant. Because wave growth is limited by the finite amount of moisture in the mean state that can be converted into heating, the unstable wave must eventually stop growing when it reaches the amplitude at which the moisture supply is exhausted (i.e., when perturbation precipitation amplitude equals mean precipitation)—an amplitude which is independent of initial growth rate. At that amplitude, we suggest that the wave can adjust or equilibrate by storing the excess moisture and converting only a fraction $(1/R'=1/\alpha)$ of the water vapor into latent heat release. The wave thus acts as a self-consistent, neutrally forced wave. This equilibration process can only occur if the wave can be self-maintained, i.e., "growing" in the instability analysis or with $R'>1$ and phase lock in the forcing calculation. As we have seen, the two conditions are closely related since the two types of solutions are just analytic continuations along the $\lambda_n(\alpha)$ curves.
Implicit in the above is the assumption that moisture storage is in phase with convergence and precipitation. To allow for the more general case of a phase difference would necessitate an explicit time-dependent moisture budget equation and a more detailed boundary layer treatment. Such an extension is beyond the scope of this investigation.

If the storage of moisture is a small fraction of the wave convergence—say, less than a third—then the structure of the moist layer can still be considered unperturbed by wave activity, as originally assumed in the moisture budget. Reed and Recker's (1971) observational analysis of synoptic disturbances indicates that the storage term is actually only 30% of the convergence term (and also, unfortunately, not in phase with the convergence). Other wave scales would then not "see" the effects of this equilibrated wave. Hence the different waves could grow to finite amplitude independent of each other. The growth of one wave does not preclude the growth of others. Therefore, one would not expect to observe only the "fastest growing" mode, and growth rate appears to be an unsuitable parameter for predicting wave occurrence. Most important, synoptic-scale disturbances can coexist with fast squall lines.

It should be emphasized that the moisture convergence must be consistent with the parameterized heating for a wave-CISK solution. In the present forcing calculation, this means that only those forced responses for which the phase of the response \( w(z_p^e) \) equals the phase of the heating parameter \( w_q^e \) represent wave-CISK. Any deviation in phase indicates that a wave with that prescribed set of scales cannot be maintained indefinitely by latent heat release. In nature, adjustment of the boundary layer (e.g., through storage) can cause phase difference between convergence and heating, effectively making \( \alpha \) complex; in this study we make the simplifying assumption that \( \alpha \) is real.

4. Specification of the heating profile

A fully closed parameterization theory would define the structure of the heating in terms of the large-scale wave fields themselves. Stark (1976) has used the Arakawa and Schubert (1974) parameterization for just this purpose in a wave-CISK calculation. As discussed in SLS, we consider this scheme to be of questionable validity. Therefore, we prescribe the shape of the heating function, keeping it the same for all wave scales; no feedback is allowed by which the wave fields can affect the heating profile. In contrast with earlier inviscid studies, SLS found that wave response was relatively insensitive to the exact shape of the heating—a direct result of the mixing action of cumulus friction.

Since the heating is due primarily to the effect of deep clouds, which heat the entire tropospheric column, we assume there is no phase tilt of the heating function in the vertical. The vertical component of the heating can therefore be taken to be a real function; the phase of all other fields refers to the deviation from the phase of the heating. Consistent with the geometries used in this study (midlatitude \( \beta \)-plane, midlatitude \( f \)-plane and nonrotating plane), we assume a single Fourier mode for the horizontal structure and time dependence of the heating, \( \exp[i(kx + ly + \omega t)] \), where \( x, y \) are the zonal and meridional coordinates and \( k, l \) the appropriate wavenumbers.

Since the determination of a self-maintaining wave-CISK mode depends on the response in vertical velocity \( w(z_p^e) \) at the top of the moist layer, it is reasonable to inquire whether any special constraints on the heating shape should be imposed, especially at \( s_c^e \). In particular, should the value of the heating at \( s_c^e \) have any relation to the heating parameter \( w_q^e \)?

Some guidance can be derived from a very simple parameterization which explicitly relates convergence and heating. If we assume the cloud field is composed only of non-entraining deep clouds and neglect radiation and reevaporation of detrainment liquid water on the wave scale, the integrated heating function can be written

\[
\int_0^\infty \bar{p} c_p Q' dz = \int_{s_c^e}^{s_T} \left( M_c^e - \bar{M}_c^e \right) ds' = M_c^e (s_T - s_c^e) + \bar{M}_c^e (s_T - s_c^e),
\]

where primes refer explicitly to perturbed (wave) quantities in this paragraph only; \( s \) is the dry static energy \((c_p T + g z)\). For observed disturbances, the exceedingly small perturbation temperature and geopotential make the second term negligible compared to the first term on the right-hand side. The vanishing buoyancy of cloud parcels at cloud top \( (s_T^e) \) and the conservation of moist static energy \((h = s + L_v q)\) in the non-entraining clouds give

\[
s_T^e \approx h_T^e \approx h_T^{\text{saturation}} = h_{\text{cloud}} = s_c^e + L_v q.
\]

Combining the heat and moisture budgets of Section 2 [Eqs. (2.2) and (2.4)], we have

\[
M_c^e L_v q = M_c^e (s_T^e - s_c^e) = \int_0^\infty \bar{p} c_p Q' dz = \int_{s_T^e} L_v (\bar{p} w') dq
\]

or

\[
M_c^e = \bar{p} w' q_c^e,
\]

which relates the perturbed cloud mass flux \( M_c^e \) to the heating parameter \( w_q^e \). The linearized thermal equation

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2 Strictly speaking, short waves might see the longer waves as part of the (local) mean conditions; however, equilibration can still occur independently for the different scales.
can be written
\[
\frac{d\tilde{T}'}{dt} = \frac{1}{c_p} \frac{d\tilde{w}}{ds} = \frac{1}{c_p} \left[ M_c - \tilde{p} \tilde{w}'(z) \right] \frac{d\tilde{s}}{ds} = \frac{1}{\tilde{p} c_p} \left[ \tilde{p} \tilde{w}'(z) \right] \frac{d\tilde{s}}{ds}.
\]

where \( \tilde{d}/dt = \tilde{a}/\tilde{u} + \tilde{U}(\tilde{a}/\tilde{z}) \). For this simple heating profile, we see that a neutral forced wave \( w'(z_0) = w_0' \) will have vanishing perturbation temperature at moist layer top \( z_0 \). In Appendix A, we show that this property pertains also to more general cloud models.

The approximate balance between heating and adiabatic cooling is found at all vertical levels of the synoptic-scale disturbances observed by Reed and Recker (1971). Since these disturbances propagate across the western Pacific with little change in amplitude, we suggest that such a totally compensated wave \( (d\tilde{T}/dt = 0) \) be considered neutral with respect to the wave-CISK instability. Because the wave-CISK condition is applied at the top of the moist layer \( z^*_0 \), this constraint sets the magnitude of \( Q_1(z_0) \).

Using the observation that vertical velocity \( w \) is nearly always proportional to \( \ln P \theta \) velocity \( (w^*) \) for hydrostatic waves in the fluid interior [with the exception of external Lamb waves, treated by Lindzen and Blake (1972)], we may write \( w = w^* \tilde{H} \), where \( \tilde{H} = R T / g \) is the local scale height. Applying the moist budget statement (2.2), the heating at \( z^*_0 \) has the value

\[
Q_1(z_0) = w^*_0 \tilde{H} \Gamma = \frac{\tilde{q}(\tilde{p} w_0^*)}{\tilde{q}(\tilde{p} H_0^*)} = \frac{P^*}{\tilde{q}(\tilde{p}/g)} \frac{P}{\tilde{q}(\tilde{p}/g)} \exp \tilde{z}_0.
\]

In nature, the heating rate is a smooth function of pressure. We have chosen a functional expression for \( Q_1 \) which is smooth and which also allows us to specify the level of maximum heating \( P_{\text{max}} \).

\[
Q_1 = \begin{cases} 
(b(p - p_T)(p - p_2)(p - p_0), & p_T \leq p \leq p_q \\
Q_1(z_0) \left( \frac{p - p_T}{p_c - p_q} \right)^a, & p_q \leq p \leq p_c \\
0, & \text{elsewhere.}
\end{cases}
\]

Note that \( Q_1 \) is continuous (i.e., zero) at the boundaries of the heating region \( z^*_0, z^*_2, z^*_T \). Requiring \( Q_1 \) to be continuous and smooth (i.e., its first derivative continuous) in the interior of the cloud layer and specifying \( P, P_{\text{max}}, \Gamma \) and \( \tilde{q} \) then determines all the parameters of the heating profile. In addition to the previously given thermodynamic variables \( \Gamma \) and \( \tilde{q} \), we use \( P = 1.2 \) cm day\(^{-1} \) and \( p_{\text{max}} = 480 \) mb to approximate the observed heating profiles. The resulting profile is displayed in Fig. 10 of SLS.

5. Wave-CISK without cumulus friction

Before proceeding with the quasi-instability calculation including the important cumulus transport of momentum, it is instructive to first consider an analogous calculation, differing only in the neglect of this physical process, in order to discover how cumulus friction changes the stability properties. In this section the results of such a preliminary calculation are presented. Our results are consistent with previous studies, given the major differences in methodology, parameterization of a moist layer that extends to a level \( z^*_e \) within the deep cloud layer, and the explicit use of a moisture budget. Other than the cumulus friction, the model parameters (e.g., heating profile, moisture budget and subcloud layer structure) are assigned the same values as in the earlier computations.

As already discussed, the key parameter for studying wave-CISK with a forcing-type calculation is the amount of moisture converged by the wave relative to the specified amount of moisture precipitated, represented by the complex ratio

\[
R' = \frac{w(z^*_0)}{w_0}.
\]

The wave-CISK consistency condition is that the phase of the response \( w(z^*_0) \) match the phase of the forcing \( w_0 \), i.e., phase \( (R') = 0 \). When this condition is met, the ability of the wave to maintain itself is measured by the magnitude of \( R' \). In the following figures, the amplitude is given as the percentage excess of the response relative to the forcing, viz.,

\[
A' = (|R' - 1|) \times 100\%.
\]

Thus an amplitude \( A' = 0\% \) is neutral \((R' = 1)\), \( A' < 0\% \) indicates stability \((R' < 1)\), and \( A' > 0\% \) indicates instability \((R' > 1)\)—when phase \( (R') = 0 \).

If the surface drag coefficient is set to zero, the instability is wave-mode-independent in the sense that the equivalent depth \( h \) (or vertical wavelength \( L_e \), which is approximately proportional to \( \sqrt{h} \)) is the only wave parameter in the calculation. Fig. 3 displays the response \([A' \text{ and phase } (R')]\) as a function of the imposed wave parameters, represented by vertical wavelength. The longest wavelength wave-CISK solutions \([\text{with phase } (R') = 0]\) are at \( L_e = 8.8, 4.4, 1.9, 1.24 \) km with amplitudes \( A' = +60, -20, +20, -13\% \), respectively. As \( L_e \) increases, \( A' \) approaches \(-100\% \) and phase approaches \(-180^\circ \). For vertically trapped modes (not shown), the phase is \( 0^\circ \) and the amplitude is stable for all vertical \( e \)-folding scales, with the amplitudes approaching neutrality as the scale decreases (i.e., as the wave energy is locally trapped on a shorter scale). For small vertical wavelengths, phase-locked solutions appear closer and closer to neutrality from both the stable and unstable sides as the wavelength decreases.

These solutions have a substantially different character from the forced solutions of Section 3 with a
discontinuous heating profile. There the mode of largest amplitude was the third mode in order from long to short wavelengths; here with a smooth heating profile, the longest mode \( L_s = 8.8 \text{ km} \) is most unstable. For smooth, “realistic” heating profiles, the vertical scale of largest amplitude is rather insensitive to the (assumed) depth of the moist layer; if the phase condition is applied at cloud base \( z_\text{c} = 0.07 \) rather than at \( z_\text{c} = 0.25 \) and the same smooth heating structure is used, then the wavelength of largest amplitude is again around 9 km, virtually unchanged.

For “realistic” heating profiles, Chang (1976) found all modes with shorter wavelengths than the first \([0(10-20 \text{ km})]\) to be stable with respect to wave-CISK, with greater stability as the vertical wavelength decreased. By contrast, the shorter wavelengths in this section tend to neutrality. This disparity is easily understood in light of the forcing calculation. For smooth heating functions, Green (1965) showed that the vertical velocity tends to follow the heating function (i.e., compensation of diabatic heating by the adiabatic cooling) as the vertical wavelength decreases relative to the scale of heating. Assuming this directly forced “particular” solution can approximately satisfy appropriate boundary conditions (as the small-wavelength solutions in fact do), then only a very small component of the “free” solution with the scale of the vertical wavelength will be excited. By the constraint imposed on the heating function, these compensated solutions (at \( z_\text{c} \)) are required to be neutral. On the other hand, Chang’s consistency condition on \( w \) is imposed at a subcloud level \( (z_\text{b}) \) below the heating region. In the

forced analog to Chang’s instability calculation, decreasing wavelength implies greater compensation and \( w(z_\text{b}) \to 0 \). The forced response \( w(z_\text{b}) \) is therefore much less than the convergence parameter \( w_\text{b} \) assumed in the heating. Thus as wavelength decreases, the modes become increasingly stable because the heating cannot maintain a low-level convergence field which is necessary for wave-CISK.

We briefly consider here the effect of surface drag on the instability of an otherwise inviscid wave. With nonzero drag coefficient \( D \) (see SLS for a simple parameterization of \( D \)) the wave response depends on the horizontal scales and time scales in a way which cannot be summarized in a single parameter. Therefore, the horizontal scales, zonal wavelength \( L_x = 2\pi/k \) and meridional wavelength \( L_y = 2\pi/l \) must be specified when considering the effects of surface drag.

The continuous lines in Fig. 4 depict the response \( w(z_\text{b}) \) to long-period forcing with \( L_x = L_y = 4000 \text{ km} \) and \( D = 2 \times 10^{-5} \text{ s}^{-1} \) on a midlatitude \( \beta \)-plane. These solutions are internal Rossby waves, for which the inviscid vertical wavelength decreases as the wave period increases. The response curves are qualitatively similar to the inviscid result; however, the features are generally displaced to longer wavelength (shorter period). Phase lock solutions are at \( L_x = 17, 5.9, 2.1 \) and 1.25 km with amplitudes +68, -30, +20 and -12%. These amplitudes are essentially identical to the no-drag case, but the wavelengths of the lowest modes are substantially increased. The possibility of a strong unstable solution at long wavelength is indicated by the amplitude bulge at \( L_x = 50 \text{ km} \); for other wave parameters the zero phase crossing could conceivably occur.
in the vicinity of the bulge, indicative of a very unstable response. As a general rule, internal Rossby waves are most unstable at long vertical wavelengths (i.e., short periods), and relatively neutral at small vertical scales (i.e., long periods or essentially advected disturbances with \( c = \bar{U} \)).

The \( \times \) symbols on Fig. 4 indicate the response of a wave in which all wave parameters are unchanged but the drag coefficient is reduced from \( 2 \times 10^{-4} \) s\(^{-1} \) to \( 1 \times 10^{-5} \) s\(^{-1} \). The Rossby wave solutions are quite insensitive to the exact value of the drag coefficient so long as its order of magnitude is appropriate for the observed tropical atmosphere.

Unlike the case with Rossby waves, surface drag has essentially no effect on the stability properties of internal gravity waves with time and space scales typical of observed squall lines. For these disturbances the intrinsic time scale \( (\Delta^{-1}) \) is much shorter than the time scale associated with surface drag \( (D^{-1}) \).

These results concerning the effect of surface drag on Rossby and gravity waves in an otherwise inviscid atmosphere are implicit in the work of Shapiro (1977).

6. Low-frequency waves with realistic cumulus transport of momentum

Wave-CISK is investigated for low-frequency waves which have dispersive properties characteristic of internal Rossby-type waves. Throughout this section we use a constant magnitude for the mean cloud mass flux, \( gM_{c0} = 5 \) mb h\(^{-1} \), and the standard profile of SLS for the mass flux. Latitude 9\(^\circ\)N is chosen as the reference for the equivalent midlatitude \( \beta \)-plane, as discussed in SLS.

Fig. 5 shows the wave-CISK response \( R' \) to long-period forcing with horizontal wavelengths \( L_x = L_y = 4000 \) km. The only change from the calculation of Fig. 4 is the inclusion of cumulus friction. The ordinate \( L_z \) is the vertical wavelength corresponding to the variable frequency \( \Delta \) and specified horizontal wavelengths \( k, l \) through the inviscid dispersion relation

\[
\left( \frac{2\pi H_0}{L_z} \right)^2 = \frac{R'}{g h} = \frac{\Delta}{4 \omega} \left( \frac{k^2 - \omega_p^2}{k^2 + \omega_p^2} \right)
\]

where \( H_0 = \) the scale height at the surface, \( R' = \) the gas constant and \( \omega_p = \beta k / (k^2 + \omega_p^2) - \Delta \). With significant cumulus friction, \( L_z \) has no physical meaning in the cloud layer, where the dynamics are governed by a fourth-order vertical structure equation; it remains, however, a convenient parameter. Note that both the amplitude and phase scales of Fig. 5 have been magnified by a factor of 10.

The profound influence of cumulus friction is evident. The amplitude excess is limited to a very narrow range of only a few percent from neutrality. Phase of moisture convergence consistently lags behind the heating by a few degrees, with no phase-lock solutions. The Doppler-shifted rate of temperature change at the trade inversion is just the small residual difference between diabatic heating \( (\propto \omega) \) and adiabatic cooling \( [\propto \omega(z_0^*)] \). Its phase is also depicted in Fig. 5; it is approximately in quadrature with the heating, indicating that the perturbation temperature of these calculations is in phase with the heating (warm-core).

The time and space scales of the Reed and Recker (1971) wave (Doppler-shifted period \( \tau = 2\pi / \Delta \approx 15 \) days, \( L_x = L_y = 4000 \) km) correspond to a vertical wavelength of \( \sim 10 \) km in Fig. 5. The phase lag of the wave-CISK parameter \( R' \) was found to be insensitive to variation of mixed layer, heating and cloud mass flux parameters. When the drag coefficient, level of maximum heating and shape of the mass flux profile were changed significantly, phase \( (R') \) ranged only between \(-9^\circ\) and \(-14^\circ\). Thus we expect our conclusions to be valid irrespective of the precise choice of these specified parameters.

The search for wave-CISK solutions must extend over a variety of horizontal space scales as well as over time scale. The results of such a search are displayed in \( L_x-L_y \) space for a fixed value of \( L_y \). For a given \( L_x \) and \( L_y \), we consider those periods which would correspond to a vertical wavelength in the range \([3, 100] \) km when there were no cumulus friction. Thus, in Fig. 6 the region between the two solid lines represents those periods and zonal wavelengths for which the inviscid dispersion relation yields vertical wavelengths in the range \( L_z \approx [3, 100] \) km when the meridional wavelength \( L_y = 4000 \) km. This meridional scale is chosen as characteristic of observed tropical disturbances (cf. Yanai et al., 1968; Nitta and Yanai, 1969). The abscissa in Fig. 6 is the zonal wavelength \( L_z \) with units \([1000 \) km],
on a logarithmic axis. The ordinate is the Doppler-shifter period $\tau$ in days, again on a logarithmic scale.

The dashed lines in Fig. 6 are isopleths of phase ($R'$) in degrees. For $L_x > 1000$ km, there is no wave-CISK solution. This behavior contrasts sharply with inviscid wave-CISK, in which an unstable solution of vertical wavelength of $\sim 10$ km was present, as well as a stable 4 km solution. An inviscid 10 km solution would be represented by a zero-phase line approximately midway between and parallel to the solid lines. Incorporating surface drag in an otherwise inviscid system changes the shape and position of the solution curves. Cumulus friction eliminates the solution altogether in this range of frequency.

When inviscid wave-CISK was treated as a forcing problem, the phase of $w(z)$ varied through a wide range of angles, both positive and negative, in the 3–100 km wavelength range. Here the phase of $w(z)$ is throughout the $L_x = 10$ region only a few degrees behind the phase of the heating. Cumulus friction causes the moisture convergence of the forced waves to lag behind the heating, never quite catching up to satisfy the wave-CISK condition.

Fig. 7 indicates the amplitude of $w(z)$ for the forced waves of Fig. 6 with $L_y = 4000$ km. Although these are not wave-CISK solutions, the amplitude of the response is of interest. In contrast to the nonfrictional cases of the last section, where amplitude excesses of 50 to 180% were commonplace in the $L_x = 10$ km range, the response does not even reach 5%. Below the 1% line the amplitude response hovers near neutrality, throughout the region within $\pm 1\%$ of the forcing.

Unlike the inviscid results (cf. Fig. 4), the amplitudes of the forced response for this and subsequent cases with a large cumulus friction term are generally at least the magnitude of the forcing. In other words, there is generally a positive excess in the response rather than a deficiency of amplitude. Therefore any perturbations that satisfy the phase condition will have sufficient amplitude for a consistent wave-CISK solution, with some excess moisture for storage in an equilibrated wave or for growth in an unequilibrated wave. Stabilization of a finite-amplitude disturbance will occur when it propagates into a region that is unfavorable for development because the phase of the converged moisture is mismatched with the phase of the precipitation, not because the amplitude of moisture convergence decays. The small phase difference of only a few degrees suggests that a finite amplitude wave will become disorganized slowly as it changes its time and space scales to adjust to the changing moisture convergence.

If we consider small-scale internal Rossby waves ($L_x < 1000$ km), the phase approaches closer and closer to zero as the zonal wavelength decreases. For example, all cases with $L_x = 600$ km have phase within $\pm 0.5^\circ$ of phase lock; with $L_x = 300$ km, $\pm 0.03^\circ$; with $L_x = 100$ km, $\pm 0.005^\circ$. Simultaneously, the amplitude approaches the forcing amplitude to within $0.1\%$. Thus, Rossby waves tend toward compensation and therefore neutrality as their zonal scale decreases. This behavior follows from a corollary of the following argument, which demonstrates why cumulus friction should reduce perturbation temperatures below inviscid amplitudes.

We now write the governing system of equations [cf. Eq. (5.10) of SLS] for the perturbation fields of vorticity $\zeta$, divergence $\delta$, geopotential $\Phi$, temperature $T$ and vertical velocity $w = dz/dt$ on a midlatitude $\beta$-plane:

$$i\omega k - f\delta + \left(\frac{gM_\infty}{p_0}\right) e^s \frac{d}{dz} \left[m(\zeta - \zeta_\infty)\right] = 0, \quad (6.2)$$

$$i\omega \delta + f\zeta + \left(\frac{gM_\infty}{p_0}\right) e^s \frac{d}{dz} \left[m(\delta - \delta_\infty)\right] = \nabla \Phi = -(k^2 + \beta)\Phi, \quad (6.3)$$
\[ \delta + \left( \frac{d}{ds^*} - 1 \right) \omega^* = 0, \]  
\[ \frac{dT}{dt} + w^\ast \Gamma = Q_1, \]  
\[ \frac{d\Phi}{ds^*} = RT. \]

Here \( m(z^*) \equiv M_c/M_{co} \) is the functional form of the mean cloud mass flux, independent of its magnitude, and \( \xi, \delta \) are the height-independent values of vorticity and divergence associated with the deep cumulonimbus clouds. As demonstrated in Table 2 of SLS, the transition from no cumulus friction \((gM_{co}=0)\) to the observed magnitude \((gM_{co}=5 \text{ mb h}^{-1})\) causes a shift in the balance of terms in the vorticity equation (6.2). With no friction, the quasigeostrophic balance between the first two terms holds; it may be written in the more familiar form
\[ \delta \frac{ds}{dt} + \beta \psi + f_0 \delta = 0 \quad \text{or} \quad -i \omega \xi + f_0 \delta = 0. \]

It is clear from (6.7) that \( \omega \) defines a time scale for the change of absolute vorticity \((\xi + f)\). The Rossby number \( Ro \equiv \omega_0/f_0 \) is much smaller than unity, guaranteeing that relative vorticity is an order of magnitude larger than divergence. In the divergence equation (6.3), the primary balance is between vorticity and geopotential terms.

With friction typical of the ITCZ, the divergence term and friction term balance in the vorticity equation; \( \tau_F \equiv (gM_{co}/\rho_0)^{-1} \) is the time required for the atmosphere to be overturned through cumulus convection. The time scales appearing explicitly in the vorticity equation are \( \omega_0^{-1} \tau_f / 2\pi \approx 5 \text{ days}, f_0^{-1} = 0.5 \text{ day} \) and \( \tau_F = 8 \text{ days} \). The effective frictional time scale \( \tau_{EF} \), however, is much shorter than \( \tau_F \), since both the exponential factor and any vertical scales less than a scale height \([\text{as in } m(z^*])\] tend to increase the magnitude of the frictional term. Hence \( \tau_{EF} \) and \( f_0^{-1} \) are comparable and faster than \( \omega_0^{-1} \). Vorticity is therefore reduced to the size of the divergence. Now the magnitude of the divergence is determined by the requirements of mass and energy conservation \([\text{as in (6.4) and (6.5)}]\), independent of the existence, parameterization or magnitude of any friction. Using the same time scales, the vorticity and frictional terms of the divergence equation (6.3) are the same size, while the \( i \omega \xi \) term is necessarily much smaller. \( \nabla \phi \) is then limited by the magnitude of \( f_0 \), but the two are not equal (as in quasi-geostrophy) with friction acting on the divergence. Since vorticity is reduced by an order of magnitude from the inviscid value, geopotential is also reduced for the same horizontal scales. The hydrostatic balance (6.6) then requires perturbation temperatures to decrease by an order of magnitude, producing the compensation between diabatic heating and adiabatic cooling in the thermal equation (6.5). This is the way in which cumulus friction acts to reduce the perturbation temperature.

We now apply this argument to the waves of shorter horizontal scale \((L_x<1000 \text{ km})\) in Figs. 6 and 7; \( \omega_0^{-1} \) is even longer than 5 days, so that both terms proportional to \( \omega_0 \) remain negligible. Therefore the only change in the vorticity and divergence equations is the pronounced increase of \((k^2 + \rho_0)\). Since \( \xi \) and \( \delta \) are already determined regardless of horizontal scale in the compensating limit \((dT/dt \ll Q_1)\), \( \nabla \phi \equiv - (k^2 + \rho_0) \phi \) retains its size and \( \Phi \) must decrease inversely proportional to \((k^2 + \rho_0)\). The dominant vertical scale is determined solely by the heating distribution in this limit; therefore hydrostaticity implies that perturbation temperature must also decrease inversely proportional to \((k^2 + \rho_0)\), enhancing the compensation and neutrality as horizontal scale diminishes.

For reduced meridional wavelength \((L_y<4000 \text{ km})\), the wave-CISK behavior is consistent with the results above for reduced zonal wavelengths: the smaller the horizontal scale, the closer is the response to compensation and neutrality. In the synoptic range of zonal wavelengths \((L_x=1000-10000 \text{ km})\), the phase of the response is still consistently negative, although much closer to 0°.

On both extrema of zonal wavelength there tends to be vertical structure of smaller scale in the thermal variables (temperature change and adiabatic cooling). This is explained by the rapid decrease of \( \omega_0 \) in both regions. The local vertical scales of our fluid system are most easily determined by a model in which slowly varying parameters of the basic state \((e.g., \text{density, temperature and cloud mass flux})\) are replaced by constants. In Appendix B we present a Boussinesq model which incorporates these simplifications and then proceed to consider the vertical scales of response in several appropriate asymptotic limits. In the present application, fixing the inviscid nondimensional wavelength \((\lambda_0 = 2\pi H_0/L_x)\) and cumulus mass flux \((M_{co})\), while allowing \( \omega_0 \) to vanish asymptotically \((\omega_0 \to 0)\), yields the following asymptotic limits for wavelength \( \lambda \) \((\text{i.e., inverse scale})\):

\[ \lambda_{1.2.3} \approx \sigma^1 \left[ \frac{(-1)i}{\sigma} \frac{\lambda_0}{\sigma \cdot Ro} \right]^i, \]

where \[ \lambda_4 \approx \frac{1}{\sigma}, \]

\[ \frac{gM_{co}}{\rho_0 \omega_0} \to \infty, \]

\[ \frac{Ro}{f_0} \to 0, \]
and

\[ \sigma R_0 = \frac{gM_{\infty}}{\rho_0 f_0} \text{ remains constant.} \]

Here \( \rho_0/g \) has been identified with the constant \( \rho_0 H \) of the Boussinesq calculation. Thus a vertical scale varying as \( \omega^4 \) is entering the solution. Precise selection of phase lock is impractical and meaningless in a numerical solution because this short scale is not resolved by the finite differencing. Phases are so close to zero, and amplitudes to neutrality, that we may consider these to be neutral wave-CISK solutions. Note the contrast to the results of two other models: 1) a dry atmosphere, in which the only "free" solution is an evanescent normal mode with a long vertical decay scale (Lamb wave); and 2) inviscid wave-CISK theories, in which at least one unstable solution arises in the 3–100 km range of vertical wavelengths, independent of horizontal scale.

As \( L_y \) is increased beyond 4000 km, the mathematical and physical assumptions used in deriving the model equations are not consistently satisfied. In particular, the \( \beta \)-plane approximation \((|\beta| \ll f_0)\) breaks down. Also, the assumption of latitudinal homogeneity for the mean fields of cloud mass flux and moisture structure on these scales is not justified. However, the qualitative trends of such calculations do provide insight for understanding the conditions under which unstable waves of low frequency can occur.

As in Fig. 6 with \( L_y = 4000 \) km, the phase lines for the \( L_y = 6000 \) km case (not shown) are everywhere negative—indicating no wave-CISK solutions. Not until \( L_y \) is increased to 8000 km, shown in Fig. 8, are there solutions with moisture convergence in phase with the heating. The position of the zero phase line is very different from inviscid theory, for it is not even approximately along lines of constant equivalent depth \( h \) or vertical wavelength \( L_y \). Rather, it is a parabolic-like curve, concave downward, roughly symmetric about the vertical line on which the zonal and meridional wavelengths are equal: \( L_z = L_y = 8000 \) km. The wave-CISK condition is satisfied for only a limited range of zonal wavelength (\( \sim 5000–15000 \) km) and for relatively long periods (\( > 40 \) days). For \( L_y = 10000 \) km, the zero phase line is similar to that of the 8000 km case.

By considering the sequence of increasing meridional wavelength, we may infer that wave-CISK instability is first realized when 1) the meridional scale is sufficiently great, 2) the zonal scale equals the meridional scale and 3) the Doppler-shifted period is relatively long. Recalling that

\[ \omega_B = \frac{\beta k}{k^2 + \delta^2}, \]

the first two points suggest that \( \beta k/(k^2 + \delta^2) \) must be sufficiently large for wave-CISK instability, and the third indicates that instability initially occurs for minimum \( \delta \). Thus wave-CISK instability can occur only when \( \omega_B \) is sufficiently large or correspondingly when the time scale \( \tau_B = 2\pi/\omega_B \) is sufficiently short.

The dashed lines of Fig. 9 give the values of \( \tau_B \) along the solution curves for \( L_y = 8000 \) and \( 10000 \) km; \( \tau_B < 6.5 \) days for all the wave-CISK solutions. By contrast, \( \tau_B \) is everywhere greater than 10 days in the \( L_y = 4000 \) km case. Wave-CISK solutions occur only when the perturbation time scale \( \tau_B \) is at least as short as the frictional time scale \( \tau_F \), so that the friction has a reduced influence on the system response:

\[ \frac{\tau_B}{\tau_F} \lesssim 1 \quad (6.8) \]
or in terms of the nondimensional number $\sigma$

$$\sigma = \frac{gM_{eo}}{\rho \omega_S} = \frac{1}{\tau_p} \frac{\tau_B}{1} \approx \frac{1}{\tau_p}.$$  \hspace{1cm} (6.9)

The solid lines of Fig. 9 give the amplitudes as a function of zonal wavelength for the $L_y = 8000$ and 10,000 km wave-CISK curves. Clearly, the degree of instability is highly correlated with $\tau_B$ or equivalently with the friction parameter $\sigma$.

In summary, we have found the following two classes of waves which satisfy the wave-CISK phase condition:

1) Neutral waves with relatively short horizontal space scales and long time scales. The friction dominates the response to convective heating, indirectly causing the temperature rate to be small. Compensation of the diabatic heating by adiabatic cooling is virtually exact, implying neutral wave response in our parameterization scheme.

2) Synoptic-scale waves for which the time scale $\tau_B$ is sufficiently short that the atmosphere is not completely overturned by cumulus convection in time $\tau_B$. Cumulus friction is still playing a role in these disturbances.

The Reed–Recker composite wave is very much affected by the vertical momentum transport associated with cumulus cloud circulations. The results of our calculation indicate that the phase of the moisture convergence lags the parameterized heating by a few degrees. With the present model conditions, it is therefore not strictly a wave-CISK mode. In a more general treatment, with mean wind shear and/or a boundary layer with less stringent assumptions on moisture storage, it could be a consistent wave-CISK solution. In any case, the near equality between the magnitude of the response and that of the forcing suggests that the wave is neither rapidly growing nor rapidly decaying, but rather "almost neutral."

7. Wave-CISK: Low-frequency waves with reduced cumulus friction

Since wave-CISK is made possible when the wave parameter $\tau_B$ is artificially reduced, we might guess that the synoptic-scale waves would become wave-CISK modes where the mean cumulus activity is weaker and the associated frictional time scale $\tau_p$ is longer. Thus if mean precipitation and cloud mass flux is half the previous strength, the condition

$$2\pi \sigma = \frac{\tau_B}{\tau_p} \lesssim 1$$  \hspace{1cm} (7.1)

can be satisfied with $\tau_B \lesssim 16$ days. This short time scale is available with $L_y = 4000$ km for zonal wavelengths near 4000 km and for long-period waves ($\tau \approx 100$ days). Much shorter meridional scales ($L_y \lesssim 3000$ km) are precluded because for them $\tau_B > 16$ days.

Explicit calculation verifies this hypothesis. Fig. 10 shows that the wave-CISK phase condition is indeed satisfied under these conditions. The solution curve with zero phase is quite localized about $L_y = 4000$ km. The minimum Doppler-shifted period is about 60 days. As shown in Fig. 11, amplitudes in this segment of parameter space have two local maxima: one in the short period range where $\tau_B$ is longest, and one in the vicinity of the solution curve where $\tau_B$ is shortest. The wave-CISK modes have amplitude excesses, indicative of instability or storage of approximately 30%.

The large amplitudes at small $\tau_B$ have already been explained in terms of reducing the nondimensional friction parameter $\sigma$ to $O(1/2\pi)$. The large amplitudes at the top of the graph, namely, for small Doppler-shifted period $\tau$, are explicable in terms of the approach to the inviscid regime. The two explanations differ according to the appropriate magnitude to which $\sigma$ is compared.

On the wave-CISK solution curves $\sigma \approx 0.2$. It is small in this absolute sense. However, it is shown in Appendix B from a simplified Boussinesq model that...
\[ \sigma_{crit} \] must be small relative to a critical value \( \sigma_{crit} \) in order to approach the inviscid regime. For the wave-CISK solutions of Fig. 10, \( L_x \approx L_y \approx 4000 \text{ km}, \quad L_z \approx 3 \text{ km}, \quad \tau \approx 60 \text{ days}, \quad Ro \approx 0.1 \). Eq. (B.12) gives \[ \frac{\sigma}{\sigma_{crit}} \approx 2 - 3 \ll 1, \] indicating that friction is still influential on the wave-CISK curve. If we keep \( L_x \) and \( L_y \) fixed but reduce \( \tau \) and thereby increase \( L_z \) [cf. Eq. (6.1)], we find that \( \sigma \) increases but \[ \frac{\sigma}{\sigma_{crit}} \] decreases. Specifically, \[ \frac{\sigma}{\sigma_{crit}} \approx 0.1 \text{ is approximately constant for small } Ro \text{ and Eq. (B12) implies that for fixed } k, l, f_0, M_{eo} \]

\[ \frac{\sigma}{\sigma_{crit}} \sim \sigma_{e0} \sim \omega_\beta^{-1} \left( \frac{\omega_\beta}{\omega_p} \right)^4 \omega_\beta^4 \sim \Delta^{-1} \sim \varphi! \] (7.2)

Therefore, the synoptic-scale wave becomes essentially inviscid if \( \tau \ll 10 - 15 \text{ days} \). This is reflected in the large amplitudes and phases typical of inviscid waves. Even though \( \sigma > 1 \), cumulus friction is not important because these waves have long vertical wavelength and are virtually barotropic: without friction, the time scale for the change of total vorticity \( (\tau) \) is very long, and no longer crucial for the wave dynamics.

We may conclude that unstable wave-CISK solutions exist on the synoptic scale in those geographic regions where mean cumulus activity is reduced by a factor of 2 from the typical ITCZ value. This leads to the statement that the instability due to latent heat release is greater where the mean cumulus activity is relatively small. Where cumulus activity is substantial, the associated momentum transport acts to eliminate both instability and neutrality (i.e., in the strictest sense of absolute phase coherence between moisture convergence and heating).

This conclusion suggests two possible situations for wave-CISK generation of synoptic waves. First, over continents the mean horizontal convergence of moisture, say by the Hadley cell or monsoonal circulations, can remain an important source of water vapor, while surface evaporation is substantially smaller than its maritime value. Therefore, the net mean cloud mass flux might be the reduced magnitude used in this section. Second, a colder ocean surface may transfer less moisture to the overlying atmosphere. A diminished evaporative moisture supply can reduce the mean cumulus activity both directly, through the local moisture budget, and indirectly, by decreasing the Hadley cell convergence (cf. Schneider, 1977).

8. Other low-frequency tropical disturbances:

**Kelvin waves**

Another class of low-frequency waves have been observed in the tropical stratosphere by Wallace and Kousky (1968). These waves have essentially no meridional velocity, decay in amplitude away from the equator, and always propagate to the east relative to the mean zonal flow. These perturbations were identified by Holton and Lindzen (1968) as a solution to the linearized wave equations on an equatorial \( \beta \)-plane. They were given the name “Kelvin waves” because of their similarity in both structure and dispersive properties with the more traditional Kelvin waves, which are edge waves trapped near a horizontal boundary on a rotating plane. The equator acts as the boundary wall for these symmetric tropical waves—precisely because of their symmetry.

Subsequently, Madden and Julian (1971) discovered another low-frequency wave in the troposphere with Kelvin-like characteristics. That oscillation also had wavenumber 1, but a longer period of 40–50 days.

In this section we investigate the effect of cumulus friction on the stability of these types of disturbances.

Kelvin waves have the strange property that they behave like gravity waves in the zonal momentum equation but satisfy a geostrophic balance in the meridional equation, i.e.,

\[ \frac{du}{dt} = \frac{\partial \Phi}{\partial x}, \quad fu = \frac{\partial \Phi}{\partial y}, \quad v = 0. \] (8.1)

For the edge waves the “zonal” direction corresponds to the direction parallel to the wall, the “meridional” direction to the transverse component. Unlike gravity and Rossby waves, equatorial Kelvin waves may oscillate with either short or long time scales. Their dispersion relation on an equatorial \( \beta \)-plane is identical to that of gravity waves on a nonrotating plane:

\[ gh = \frac{\omega^2}{k^2}. \] (8.2)

Just as was the case with equatorial Rossby modes, the inclusion of cumulus friction removes the \( y-z \) separability for equatorial Kelvin waves. In addition, the friction induces a meridional circulation. Unlike Rossby waves, Kelvin waves cannot be modeled on an infinite midlatitude \( \beta \)-plane because a horizontal boundary is required.

However, we may take advantage of their similarity to edge waves by simulating their structural and dispersive properties on a midlatitude \( f \)-plane. Inertial gravity waves have the dispersion relation

\[ \omega^2 = f_0^2 + gh(k^2 + \beta^2). \] (8.3)

Edge waves also satisfy this relation, but in the very special form

\[ \omega^2 = ghk^2 \quad \text{and} \quad 0 = f_0^2 + ghk^2. \] (8.4)

They propagate along the wall (at \( y = 0 \)) with phase speed \( (gh)^{1/2} \), but are trapped in the transverse direction \( (y > 0) \) at a scale of one (midlatitude) radius of deformation, i.e.,

\[ e^{ilv} = e^{il_0v} \quad \text{and} \quad il = -l_0i = -(f_0^2/gh)^{1/2}. \] (8.5)
Fig. 12 displays the region of $L_z = \tau$ space of interest for low-frequency Kelvin waves. We consider periods in the range 5–100 days and zonal wavelengths between 1000 and 40,000 km. The straight (solid) line identifies the points where $Ro = 1$ according to Eq. (8.7). Only in the upper right corner is $Ro > 1$. Lines of constant phase speed (not shown) are perpendicular to the $Ro = 1$ line. Mean cumulus mass flux typical of the ITCZ — $gM_\sigma = 5$ mb h$^{-1}$ as in Section 6 — has been used.

For relatively slowly traveling waves, i.e., short zonal wavelengths and long periods, represented in the lower right corner, we find that adiabatic cooling almost exactly compensates the diabatic heating, with minuscule rates of change of temperature. In particular, the response of vertical velocity at the top of the moist layer is for all practical purposes neutral. Everywhere below the two dashed curves, the phase of the moisture convergence is within $\pm 3^\circ$ and $\pm 1^\circ$ of the phase of the heating. The amplitudes are also very close to neutral in this region: below the two dot-cash lines, the amplitudes $w(z_0)$ are within $\pm 4\%$ and $\pm 2\%$ of the forcing amplitude $w_\sigma$.

This region of extreme compensation can be understood in terms of the categories of Appendix B. If we again take the small Rossby number limit of Eq. (B12), $\sigma_{\text{crit}}^2 \lambda^2 \approx \frac{2}{3}$, and also approximate $\lambda^2 \approx \frac{2}{3} \frac{R \Gamma}{(gh)}$, we find that the values of $L_z$ and $\tau$ which lie on the $\sigma = \sigma_{\text{crit}}$ line are related by

$$L_z \approx 0.01255 \frac{(10^3 \text{ km})^6}{(\text{day})^7}.$$

The dotted line displays the set of $L_z$ and $\tau$ that satisfy Eq. (8.8). For $Ro < 1$, this line is a good approximation to the boundary of neutral solutions; near $Ro = 1$, the asymptotic expansion leading to Eq. (8.8) breaks down. These neutral solutions occur in a region of parameter space that is dominated by cumulus friction ($\sigma > \sigma_{\text{crit}}$). Note also that $\sigma > (2\pi)^{-1}$ since $\tau > \tau_\text{F} = 8$ days.

In this friction-controlled region, cumulus mixing induces a meridional circulation in which the meridional wind is comparable to the zonal wind component. The zonal wind is significantly smaller than its inviscid counterpart, indicating a decrease in perturbation kinetic energy.

For $Ro < 1$ and above the $\pm 3^\circ$ curve, the phase of $w(z_0)$ is generally positive and between $3^\circ$ and $90^\circ$. The temperature rate tends to be as large as the other terms in the thermal equation. Zonal and meridional winds are again comparable, but an order of magnitude larger than those of the frictional regime for the same heating magnitude.

As the period is decreased below 1 day, Kelvin waves with typical phase speeds (between 10 and 100 m s$^{-1}$) have predominantly gravity-wave characteristics ($Ro > 1$); specifically, the response at $s = s_\text{g}$ is essentially the same as the response of gravity waves to be discussed.
in the following section. As will be shown, these waves are virtually unaffected by cumulus friction. Therefore it is not surprising that the zonal velocity is nearly an order of magnitude greater than the meridional velocity, as in the inviscid Kelvin wave.

Assuming no mean wind, the calculations of this section suggest that tropospheric waves with the time and space scales observed by Madden and Julian (1971) belong to the neutral (or perhaps slightly unstable) wave-CISK category in the frictional regime. The stratospheric Kelvin wave is not a wave-CISK mode. These conclusions are obtained with the relatively large mean cumulus mass flux of the ITZ. A possibly more appropriate, smaller value characteristic of an average over all tropical latitudes would reduce the frictional control and perhaps allow wave-CISK modes of the inviscid regime.

9. Wave-CISK at high frequencies

Another type of disturbance commonly observed in the tropics is the squall line. Squall lines have essentially two-dimensional structure and propagate horizontally with a relatively fast phase speed \( c \approx 10-20 \text{ m s}^{-1} \). Their time and space scales correspond to periods somewhat less than a day and horizontal wavelengths of several hundred kilometers. The passage of a squall line is marked by excessive rainfall and rapid changes in temperature and wind.

The scales of such phenomena are those of high-frequency gravity waves. Although observed squall lines appear to have significant nonlinear features, some insight into their behavior can be obtained from a linearized approach. Because the frequency is typically much larger than the local Coriolis parameter \( f_0 \), the Rossby number \( R_o \) is typically \( \gg 10 \). Planetary rotation therefore has only a secondary effect on these waves, and may be neglected. Setting \( f_0 = \beta = 0 \), the vorticity equation (6.2) decouples from the other governing equations. We may then define the \( x \) direction to be the direction of propagation, and neglect the \( y \) variation \( (\partial/\partial y=0) \) altogether. The linearized system of equations with cumulus friction becomes

\[
\begin{align*}
\dot{i} \omega u & = -ik \Phi + \frac{g}{\rho} \frac{d}{ds^*} \left[ M \epsilon (u-u_e) \right] \\
\dot{i} ku + \left( \frac{d}{ds^*} - 1 \right) w^* & = 0 \\
\dot{i} \omega \frac{d \Phi}{ds^*} + w^* \Gamma & = Q_1 \\
\end{align*}
\]

(9.1)

The transverse equation

\[
\dot{i} \omega v = -\frac{g}{\rho} \frac{d}{ds^*} \left[ M \epsilon (v-v_e) \right]
\]

(9.2)

and associated boundary condition is satisfied uniquely by \( v = v_e = 0 \); thus \( v \) and \( \Phi = \epsilon v_e \) vanish identically. Unlike Rossby waves, the gravity wave response is invariant with respect to the direction of phase propagation relative to the mean flow: if \( k \rightarrow -k \), then \( (u, \Phi, T, w^*, w) \rightarrow (-u, \Phi, T, w^*, w) \). This degeneracy is split by \( \beta \); the splitting is not expected to be an important effect, and has therefore been neglected in the present model.

Fig. 13 displays the phase and amplitude of \( R' \) as a function of phase speed \( c = (gh)^{1/2} \) for three different time scales: \( \tau = 0.1, 0.4 \) and 1 day. The wavelength \( L_c \) varies with the phase speed \( c \) as

\[
L_c \text{[km]} = 86.4 \times \tau \text{[day]} \times c \text{[m s}^{-1}] 
\]

(9.3)

Both phase and amplitude with \( \tau = 0.1 \) day are virtually identical to the cases shown in Fig. 3 with no cumulus friction. With a longer period \( \tau = 0.4 \) day, the results simulate the frictionless curve for \( c \gtrsim 2-3 \text{ m s}^{-1} \); for very slow wave speeds, \( c < 2 \text{ m s}^{-1} \), both phase and amplitude exceed vanish—indicating neutral wave-CISK modes. For the longer period \( \tau = 1 \) day, the same qualitative behavior is observed, but \( c \approx 5 \text{ m s}^{-1} \) divides the two regions.

All of these results follow from a consideration of time scales and nondimensional parameters introduced earlier. First, surface drag has almost no effect because its time scale is longer than any of the wave time scales here: \( 2\pi/\omega \approx 4 \text{ days} \). Second, these solutions do not resemble the unstable wave-CISK solutions of Section 7 because

\[
\frac{2\pi \sigma}{\tau_p} = \ll 1; 
\]

(9.4)
\(\tau_F\) for all these cases is 8 days, and \(\bar{\tau} \leq 1\) day. Third, the key parameter that is operating to control the solution is the ratio \(\sigma/\sigma_{\text{crit}}\) that determines whether the wave scales are in the inviscid or the friction-dominated regimes. Eq. (B16) of Appendix B gives the value of \(\sigma_{\text{crit}}\) for gravity waves on a nonrotating plane. Using \(\lambda = 3R\Gamma/(gh)\) and \(gM_\infty = 5\) mb h\(^{-1}\), the condition \(\sigma/\sigma_{\text{crit}} = 1\) occurs where

\[
c[m \text{ s}^{-1}] \approx 3.9. \tag{9.5}
\]

The triangles in Fig. 13 are placed at the critical phase speeds for \(\bar{\tau} = 0.4, 1\) day. It is evident that the solutions are for all practical purposes identical to the inviscid results at much faster speeds (\(\sigma/\sigma_{\text{crit}} \ll 1\)). At slower speeds, compensation and neutrality are nearly exact and cumulus friction is the dominant process. For \(\bar{\tau} = 0.1\) day, the entire range displayed is in the inviscid regime.

We may conclude that gravity waves (i.e., squall lines) with the observed scales, \(c \approx 12\) m s\(^{-1}\) and \(\bar{\tau} \leq 1\) day, belong to that part of the parameter range in which cumulus friction is not an important process. The previous instability at \(c \approx 12\) m s\(^{-1}\) obtained with inviscid wave-CISK dynamics (Lindzen, 1974) is essentially unmodified. Cumulus friction is important at this phase speed only for long periods \(\gg 3\) days. Green's (1965) hypothesis of compensation at short vertical wavelengths indicated that a discrete set of inviscid waves with slow phase speeds would be almost neutral; with cumulus friction, the discrete set changes to a virtual continuum when the time scales are sufficiently long.

It should be noted that these conclusions depend on the very simple parameterization of convection used throughout this study. Any cumulus parameterization breaks down when the wave time scale \((\bar{\tau}/2\pi)\) approaches the cloud time scale, which is usually estimated to be of the order of 30 min to 3 h. Therefore, these results are at best only a qualitative indication of large-scale gravity wave dynamics on the very fast time scales.

10. Summary and conclusions

The model of SLS is implemented to attack the problem of understanding why the synoptic-scale waves are in fact observed in the tropics. By relating the diabatic heating to the convergence of moisture through a simplified moisture budget, a consistency condition is introduced to calculate what wave scales are self-induced and self-maintained by the wave-CISK process. For the synoptic-scale waves considered in SLS, the amplitude of moisture convergence is consistent with the amplitude of heating, but the phase lags by a few degrees. In the context of the model assumptions, these waves are not locally self-generated. However, the small phase difference suggests that a wave which is generated elsewhere and then propagates into the western Pacific ITCZ region will adjust to the local mean conditions by slowly changing its horizontal scale or by responding with some storage of moisture. The relatively small difference in phase between convergence and heating indicates that the wave can survive as an organized entity over several wavelengths when the mean wind shear is small. The dynamics of the adjustment are not amenable to further study with the present forcing model with fixed space and time scales.

One class of low-frequency, Rossby-type waves does appear to satisfy the wave-CISK condition on vanishing phase difference between moisture convergence and precipitation. Waves with relatively short horizontal space scales and long time scales are dominated by the cumulus friction; the temperature perturbation is indirectly forced to disappear. Hence the adiabatic cooling balances the diabatic heating everywhere in the wave. With the parameterization scheme used here, neutrality of response in both phase and amplitude is implied.

Significant instability results when the vorticity time scale \(\tau_\phi\) is (artificially) reduced to the frictional time scale over which the troposphere is completely overturned through cumulus convection. For the strong mean cumulus activity typical of the maritime ITCZ, this frictional time scale is somewhat shorter than the time scales of observed waves. However, if we consider regions of less cumulus activity (e.g., over continents or perhaps over a colder sea surface), then instability is possible for the observed synoptic time and space scales. Therefore we may conclude that the observed synoptic disturbances of the western Pacific are slowly changing waves which are generated in other geographic regions of reduced mean cumulus activity.

Two other types of waves are also observed in the tropics: low-frequency Kelvin waves and high-frequency mesoscale disturbances. In considering the effect of cumulus friction on these disturbances, we find that Kelvin waves at very long periods (~50 days) are neutralized—in spite of strong surface friction—while the fast gravity waves with short periods less than a day are virtually unchanged from those of inviscid theory.

All of the foregoing conclusions have been obtained using a linear model with constant wind and constant tropospheric stability. Both of these vary considerably in the tropics, particularly in the vertical dimension. The basic model is currently being adapted in order to simulate tropical waves with these more realistic mean conditions.

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APPENDIX A

Neutrality of Compensated Waves and Cloud Parameterization

In Section 4, we argued that our heating profiles ought to have the property that compensated waves are neutral. In such cases where we can write

$$c_p\rho Q_1 = M_c \frac{\partial T}{\partial z},$$

(A1)

the above is equivalent to requiring that

$$M'_c = \rho \omega' \text{ at } z = z_q.$$  (A2)

Relatively our criterion for stability (instability) is that in the forced problem the phases of $\rho \omega'$ and $M'_c$ agree at $z_q$ and that $|\rho \omega'| < |M'_c|$, i.e., $|\rho \omega'| > |M'_c|$. Thus the above assumption calibrates the boundary between stability and instability. Indeed, as we have noted, failure to note this “calibration” accounts for some differences in the conclusions of various wave-CISK analyses.

We have further shown for a simple model wherein the entire moist layer (which, for convenience, we refer to as being bounded by $z_q$) is completely mixed with respect to $q$, and where the clouds do not entrain above $z_q$ and detrain only at the level of zero buoyant acceleration (for air arising in the moist layer), that (A2) is automatically satisfied. The purpose of this Appendix is to show that (A2) holds for more general situations as well. We will also show that in some instances, at least, the failure of (A2) to hold arises from obvious modeling inconsistencies.

Instead of our completely mixed moist layer, consider a fluid with the following specification of $T$ and $q$:

$$T(z) = \begin{cases} T(0) - (g/c_p)z, & 0 \leq z \leq z_B \\ T(0) - (g/c_p)z + \Delta T, & z_B < z \leq z_q \\ T(0) - (g/c_p)z + \Delta T - \Gamma(z-z_q), & z > z_q \end{cases}$$

(A3)

$$q(z) = \begin{cases} q_1 \text{ (constant)}, & 0 \leq z \leq z_B \\ q_2 = q_1 - \delta q, & z_B < z \leq z_q \\ 0, & z > z_q. \end{cases}$$

Clearly, air originating below $z_B$ will achieve neutral buoyancy at a greater height than air originating between $z_B$ and $z_q$. Let $z_{T1}$ and $z_{T2}$ represent these two heights; then

$$c_p T(z_{T1}) + g z_{T1} = c_p T(0) + L \omega q_1$$

and

$$c_p T(z_{T2}) + g z_{T2} = c_p[T(0) + \Delta T] + L \omega q_2.$$  (A4)

We next consider a cloud model consisting in two types of nonentraining clouds: one associated with a mass flux $M_{c1}$ originating below $z_B$ and detraining at $z_{T1}$ and the other associated with a mass flux $M_{c2}$ originating between $z_B$ and $z_q$ and detraining at $z_{T2}$. Thus

$$M_c = M_{c1} + M_{c2},$$

(A5)

$$c_p Q_1 = M_c \left[ \frac{\partial T}{\partial z} + \frac{g}{c_p} \right].$$  (A6)

Once more, we require that total heating equal the latent heat of all evaporated and converged moisture, i.e.,

$$\int_0^\infty c_p Q_1 dz = L(\rho(z_B)w(z_B)q_1$$

$$+ \{\rho(z_q)w(z_q) - \rho(z_B)w(z_B)\} \frac{\delta q}{q_1} + E).$$  (A7)

Moreover, using (A4)–(A6) we have

$$\int_0^\infty c_p Q_1 dz = M_{c1}L \omega q_1 + M_{c2}L \omega q_2.$$  (A8)

Note that while (A7) still constrains $M_{c1}$ and $M_{c2}$, it no longer determines them. We next examine the consequences of various choices of $M_{c1}$ and $M_{c2}$. If we assume $M_{c2}=0$, then (A3) and (A7) imply

$$M_c = M_{c1} = \left\{ \begin{array}{ll} \rho(z_q)w(z_q) + \frac{E}{q_1}, & \rho(z_q)w(z_q) - \rho(z_B)w(z_B) \frac{\delta q}{q_1} \\
\end{array} \right.$$  (A9)

If, moreover, we neglect time variations of $E$, $q_1$, $q_2$ and $T$ (consistent with the main text) then the wave component of $M_c$ will be given by

$$M_c' = M_{c1} = \rho(z_q)w'(z_q)$$

$$\times \left[ 1 - \frac{\rho(z_q)w'(z_q) - \rho(z_B)w'(z_B) \delta q}{\rho(z_q)w'(z_q) q_1} \right].$$  (A10)

In general, the quantity in the brackets will be less than 1, implying stability for full compensation at $z_q$. If instead we assume $M_{c1}=0$, then we obtain instead of (A10)

$$M_c = M_{c2} = \rho(z_q)w'(z_q) \left[ 1 + \frac{\rho(z_B)w(z_B) \delta q}{\rho(z_q)w'(z_q) q_1} \right].$$  (A11)

In general $[\text{unless } \delta q = 0 \text{ in which case both (A10) and (A11) reduce to the result obtained in Section 4}]$ the quantity in brackets will be greater than 1, implying instability for full compensation at $z_q$. Of course, both of the above choices are unreasonable: one would not expect that all buoyant elements would rise to either
the height reached by the most buoyant elements or
the height reached by the least buoyant elements.
Indeed it is almost intuitively evident that the former
would tend to stabilize CISK while the latter would
lead to destabilization.

A more plausible choice would be to assume that
moisture converged below \( z_B \) rises to \( z_T1 \) while moisture
converged between \( z_B \) and \( z_q \) rises to \( z_T2 \), in which case
\[
M_{\varepsilon 1} = \rho(z_B)w(z_B) + E/q_1
\]
\[
M{\varepsilon 2} = \rho(z_q)w(z_q) - \rho(z_B)w(z_B)
\]
(A12)

In this case, we have at \( z = z_q \)
\[
M'_{\varepsilon} = M'_{\varepsilon 1} + M'_{\varepsilon 2} = \rho(z_q)w'(z_q)
\]  
(A13)

which is exactly the condition we assumed \[i.e.,
Eq. (A2)]

It is important to note that the above procedure is
uniquely generalizable to a situation where \( q \) and \( T \)
 vary continuously in the moist layer with Eq. (A2)
remaining true. One merely divides the moist layer into
a large number of layers analogous to the two layers
discussed above; the contribution of each layer to \( M_c \)
will simply be the horizontal convergence within that
layer, and that portion of \( M_c \) will detract at the appro-
priate level of zero buoyancy. In the limit of these
layers becoming infinitesimally thin, we arrive at a
model of deep clouds in which there is continuous
entrainment within the moist layer (below \( z_q \)), and
above \( z_q \) there is continuous detrainment over all those
levels which constitute zero buoyancy levels for air
below \( z_q \). Preliminary estimates suggest that this leads
to reasonable simulations of observed distributions
of \( Q_1 \) and thus lend support to the use of Eq. (A2).
The large detrainment usually observed below \( z_q \) is
certainly attributable to shallow trade cumuli and not
to deep clouds.

It should be noted that data for \( M_c \) (or \( Q_1 \)) always
refers to an ensemble of clouds. It is not clear that the
appearance of “continuous” detrainment implies de-
trainment which is continuous or an ensemble of clouds
with different detrainment levels; it seems unlikely that
such distinctions are relevant to cloud parameterizations.

APPENDIX B
Vertical Response Scales with Cumulus Friction

An approximation to the local vertical scales of
response for waves in an atmosphere with cumulus
friction can be obtained by treating the same physics
in a stratified Boussinesq fluid. If the cumulus mass
flux is assumed independent of height in the cloud
layer, the governing differential equations then have
constant coefficients; the complex vertical wave-
numbers are the roots of a quartic polynomial. These
vertical scales provide an aid for understanding wave
behavior in terms of the various nondimensional
parameters which relate the time and space scales of
the wave and the strength of the mean cumulus activity.
For example, inviscid-like behavior can be expected
when the friction is sufficiently weak that two of the
wavenumber solutions approach the inviscid wave-
numbers. The unrealistic assumptions of constant
density, stratification and mass flux disallow any
application of quantitative wave response to diagnostic
and instability calculations. However, the estimation
of vertical scales is quite useful for determining solution
regimes in terms of the relevant parameters.

We consider linearized hydrostatic perturbations in a
Boussinesq fluid for which the density is assumed
constant (\( \rho_0 \)) in the governing equations, except where it
appears with the gravitational acceleration \( g \) in the static
stability \( N^2 = -(g/\rho_0)(d\rho_0/dz) \). We further
assume that the mean cloud mass flux is constant with
height (\( M_c \)) in the cloud layer. Under these conditions,
and with the midlatitude \( \beta \)-plane geometry, the model
equations in \( z \)-coordinates are
\[
\begin{align*}
\frac{i\omega\Phi - f_0 x}{\rho_0} + \frac{M_c}{\rho_0} \frac{d\psi}{dz} &= 0 \\
\frac{i\omega\Phi}{\rho_0} + \frac{M_c}{\rho_0} \frac{dx}{dz} &= \Phi \\
\frac{d\Phi}{dz} + \frac{gQ}{\rho_0} &= 0 \\
\frac{dw}{dz} &= 0 \\
\frac{dx}{dz} &= 0
\end{align*}
\]  
(B1)

where \( \Phi \) is now the perturbation pressure divided by
the mean density, \( w \) the vertical velocity and \( Q \) an
effective mass sink; \( \psi \) is the streamfunction and \( x \) the
velocity potential, which together define the horizontal
wind. The cloud variables \( \psi \) and \( x_c \) enter explicitly
through the boundary conditions but do not appear in
the governing equations when the cloud mass flux
is constant with height. This system can be compactly
written in matrix form
\[
\begin{bmatrix}
M_c/\rho_0 & 0 & 0 & 0 \\
0 & M_c/\rho_0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & i\omega
\end{bmatrix}
= \begin{bmatrix}
\psi \\
x \\
w \\
\Phi
\end{bmatrix}
\]

(B2)

The solution will consist of five vertical scales: the
scale of the forcing \( Q \) and the four “free” response scales.
For a smooth forcing function the forced scale is on the order of the depth of the forcing region (e.g., the deep cloud layer of the tropical wave-CISK problem). The other response scales, which are generally excited when the forced solution per se does not satisfy the appropriate boundary conditions, are obtained from the homogeneous equation solutions. Since all coefficients on the left are constant by assumption, the four free solutions are exponential functions of the form \( \exp(i\lambda z/H) \), \( j=1, 2, 3, 4 \); \( H \) is an appropriate vertical depth scale for the fluid. The vertical response scales are then \( H/\lambda_j \) and oscillatory modes have wavelength \( 2\pi H/\lambda_j \).

Replacing \( d/dz \) with \( i\lambda/H \) and requiring the determinant of the coefficient matrix to vanish yields the characteristic equation for \( \lambda \):

\[
\begin{vmatrix}
1 & \frac{M_{20}}{\rho_0} & \frac{\lambda}{H} & 0 \\
0 & 1 & -\frac{\lambda}{H} & 0 \\
0 & 0 & N^2 & (i\phi) \frac{\lambda}{H} \\
0 & 0 & 0 & 1 \\
\end{vmatrix} = 0.
\]  

(B3)

This quartic equation can be expressed in nondimensional form in terms of three nondimensional parameters as

\[
\lambda^2(\lambda+2) \left( \frac{R_0^2}{1-R_0^2} \right) - \lambda^2 + \lambda^2(\lambda+1) = 0.
\]  

(B4)

As before, \( R_0=\omega_f/f_0 \) is a Rossby number which is the ratio of the time scale of the Coriolis acceleration \( (f_0)^{-1} \) to a time scale of the wave \( (\omega_f)^{-1} \); \( \sigma=(M_{20}/\rho_0)/H\omega_f \) is a measure of the intensity of the cumulus convection, i.e., it is the ratio of the same time scale \( \omega_f^{-1} \) of the perturbation to the time required for all the fluid in the depth \( H \) to be convectively overturned through the deep cumulus clouds; and \( \lambda_\sigma^2 \) is the square of the nondimensional vertical wavenumber when cumulus friction is absent, i.e.,

\[
\lambda_\sigma^2 = \frac{\omega_f^2}{\rho_0 f_0^2 - \omega_f^2} = \frac{H^2 N^2}{gh}.
\]

(B5)

Note that the characteristic equation reduces to \( \lambda^2 = \lambda_\sigma^2 \), when \( \sigma=0 \). The Doppler-shifted wave frequency \( \phi \) appears explicitly only in \( \lambda_\sigma^2 \), while \( \omega_f \) is explicitly contained in all three parameters.

In SLS we examined the system response as the intensity of the cumulus friction was increased while keeping the inviscid wave parameters fixed. Fig. 14 shows the wavelengths and \( \epsilon \)-folding scales for waves in a Boussinesq fluid under similar circumstances. Here \( f_0 \) corresponds to \( 9^\circ \text{N latitude}; R_0=0.11 \); the inviscid vertical wavelength is \( 10 \text{ km}; H=8.8 \text{ km}; N^2=0.8 \times 10^{-4} \text{s}^{-2}; \) and \( \sigma=0.6 \) corresponds to a cloud mass flux of approximately \( 5 \text{ mb h}^{-1} \), roughly the same as the mean mass flux observed by Yanai et al. (1973).

Wavelengths \(-L_2 \) and \(-L_4 \) asymptote to \( L_0=10 \text{ km} \) in the limit \( \sigma \rightarrow 0 \). The asymptotic expansion of Eq. (B4) in this limit, for fixed \( R_0 \) and \( \lambda_\sigma^2 \), yields

\[
\lambda_{\sigma=0} \sim \pm \lambda_0 \frac{-1+R_0^{-1}}{\sigma}.
\]

(B6)

Two of the roots, represented by \( L_1 \) and \( L_4 \), tend toward increasingly smaller scale as \( \sigma \rightarrow 0 \); they are independent of \( \lambda_0 \), to lowest order in \( \sigma \). These free solutions preclude the numerical solution of the general non-Boussinesq system for very small magnitudes of cumulus friction, necessitating the introduction of some eddy viscosity to smooth the solution. In the limit of large cumulus friction (\( \sigma \rightarrow \infty \)), all four scales are asymptotically increasing, but one much faster than the other three:

\[
\lambda_{\sigma=\infty} \sim -\sigma^{-1}, \quad (1-\left[ \lambda_0^2 \left( \frac{1}{R_0^2} - 1 \right) \right]^{-1}) \sigma^{-1}.
\]

(B7)

Three of the four roots in this limit result from the three cube roots of \(-1 \), two of which are complex conjugates. The other two roots are real, indicating propagating modes.
The transition region between the two asymptotic regimes occurs exactly in the parameter space corresponding to tropical large-scale waves \((0.1 < \sigma < 1)\). The transition in vertical propagation characteristics suggests that the wave fields will respond to forcing very differently from the behavior of the inviscid, small \(\sigma\) limit; indeed, the magnitudes of the velocity and temperature fields shown in Fig. 9 of SLS display a sharp change in response at just this point. For small convection magnitude, all four modes are oscillatory. As \(\sigma\) increases to and then beyond a wave-parameter-dependent critical value (in Fig. 14, \(\sigma_{\text{crit}} = 0.28\)), two of the roots converge and then bifurcate into a complex conjugate pair. The wave response can thus be expected to change: the influence of the cumulus transport of momentum propagates to other vertical levels with small \(M_{\text{c0}}(\sigma < \sigma_{\text{crit}})\), but is trapped for two of the response modes with larger \(M_{\text{c0}}(\sigma > \sigma_{\text{crit}})\). As in the parameterization of friction by a linear Rayleigh coefficient, solution magnitudes are damped when the frictional time scale is comparable to the wave time scale. Significantly unlike the case of Rayleigh friction, two other propagating modes of response are available to the fluid system and the transition is quite sudden.

For moderate values of \(\sigma(\sim 1)\), the trapped modes are highly localized, with \(\epsilon\)-scaling scales as short as 0.84 km. It is interesting to note that these two evanescent modes are continuations of one of the inviscid-like solutions and one of the singular (for small \(\sigma\)) solutions. If the moisture availability and associated instability allowed the cumulus activity to grow further, the response scales would all increase, either as \(\sigma^1\) or \(\sigma^1\). As these scales approach the vertical scale of the forcing, the amplitudes of the dynamic fields resume a linear response to the forcing magnitudes.

The intermediate region in \(\sigma(\sim 0.3)\) where the wave response changes character is not accessible to asymptotic expansion in \(\sigma\). However, we can take advantage of the small Rossby parameter to obtain the leading behavior of \(\lambda\) as \(Ro \rightarrow 0\). The expression \(B12\) for \(\sigma_{\text{crit}}\) is correct to the second term in \(Ro\) expansion and therefore provides a reasonable approximation when \(\vert Ro \vert \ll 1\); with it we can predict the transition to a friction-dominated response.

First we rewrite Eq. \((B4)\) by introducing \(\theta = \sigma \lambda\) and \(\theta_0 = \sigma \lambda_0\):

\[
\theta_0^2[1 - \theta_0^3(\sigma + 1)] - \theta_0^3(1 - \theta_0^3)(\sigma + 1) = 0. \tag{B8}
\]

Formal dependence on \(\sigma\) is thereby eliminated. To lowest order in Rossby number, the four solutions are

\[
\theta_{\text{Ro-0}} \sim \frac{1}{\theta_0^3} \left[ 1 \pm \left(1 + \frac{4}{\theta_0^3}\right)^{-1} \right], \quad \frac{1}{Ro}, \tag{B9}
\]

where we have assumed \(Ro^2 \ll \theta_0^2\) in a two-parameter expansion. For sufficiently small \(\theta_0(\theta_0^2 < 2)\), these asymptotic expressions correspond precisely to those derived from the \(\sigma \rightarrow 0\) limit \([\text{Eq. (B6)}]\) to lowest order in \(Ro\).

Fig. 14 shows that the \(L_4\) root \((\theta \approx 1/\text{Ro})\) is considerably distant from its linear asymptote where it intersects the \(L_3\) root, although extrapolation of the \(L_3\) and \(L_4\) asymptotes given by Eq. \((B9)\) would appear to give a good approximation. Specifically, according to Eq. \((B9)\) the \(L_3\) and \(L_4\) asymptotes intersect where

\[
\theta_0^3 \approx \frac{1}{\text{Ro}(1 + \text{Ro})}, \tag{B10}
\]

or \(\sigma_{\text{crit}} = 0.52\) for Fig. 14. A better approximation is obtained by carrying each expansion to the next higher power in \(\text{Ro}\):

\[
\theta_0^3 \sim \frac{1}{\text{Ro}} \left[ 1 \pm \left(1 + \frac{4}{\theta_0^3}\right)^{-1} \right] + 0.5 \cdot \text{Ro} + O(\text{Ro}^2) \tag{B11}
\]

Equating \(\theta_3\) and \(\theta_4\) to \(O(1)\) yields

\[
\theta_0^3 \approx \frac{2}{3 \text{Ro}} \left( 1 + 2 \text{Ro}(1 - \text{Ro})^4 \right), \tag{B12}
\]

or \(\sigma_{\text{crit}} = 0.40\) for the parameters of Fig. 14, slightly higher than the exact quartic solution shown in Fig. 14 but in the right ballpark.

In Section \(\delta\), we consider the limiting behavior as \(\omega_\delta \rightarrow 0\) for fixed cumulus friction magnitude \(M_{\text{c0}}\), vertical wavenumber \(\lambda_0\) and Coriolis parameter \(f_0\). Since \(E = \sigma \text{Ro}\) is the parameter (independent of \(\omega_\delta\)) which is held fixed in this limit, \(E/\sigma\) is formally substituted for \(\text{Ro}\) in Eq. \((B4)\) for an alternative three-parameter expansion

\[
E^2 \sigma^2 \lambda^2 (\sigma \lambda + 2) + (E^2 - \sigma^4) \tag{B13}
\]

The singular perturbation expansion as \(\sigma \rightarrow \infty\) for finite \(E\) yields

\[
\lambda \sim -\sigma^{-1}, \quad [-\lambda^2 E^{-2}] \sigma^1. \tag{B14}
\]

The elimination of \(\text{Ro}\) rather than \(\sigma\) is arbitrary: the same behavior of the roots for decreasing \(\omega_\delta\) results if \(\sigma\) is everywhere replaced by \(E/\text{Ro}\) and we take the limit \(\text{Ro} \rightarrow 0\). The desired conclusion is that increasingly shorter scales \((\lambda \sim 0, \omega_\delta^1)\) result as \(\omega_\delta \rightarrow 0\) with other parameters held constant.

The faster time scales and smaller space scales characteristic of gravity waves and squall lines may also be considered with the system of Eqs. \((B1)\). Typically \(\Delta^2 \gg f_0^2\) so that the appropriate limits are \(f_0 \rightarrow 0\) and \(\omega_\delta \rightarrow -\omega\); i.e., gravity-type waves on a nonrotating plane. The decoupling of the vorticity equation from the other three in this limit is represented
by the factorization of the characteristic equation, which may be expressed as

$$\left( \theta + 1 \right) \left[ \theta (\theta + 1) - \theta \phi \right] = 0. \quad (B15)$$

This equation follows formally from Eq. (B8) in the limit $Ro \to \infty$. The $(\theta + 1)$ term derives from the vorticity equation; there is no forcing for the vorticity, its amplitude is generally zero. In the limits $\sigma \to 0$ and $\sigma \to \infty$, the leading terms in the asymptotic expansions for $\lambda$ are identical to those of Eqs. (B6) and (B7) if $1/Ro$ is replaced by zero. The critical value of $\sigma$ which marks the transition to two complex roots is given exactly by

$$\sigma_{\text{crit}} = \left( \frac{4}{27} \right)^{1/4}$$

or

$$\left( \frac{M}{\rho_0} \right)_{\text{crit}} = \left( \frac{4}{27} \right)^{1/4} \frac{\Omega}{N} \left( gh \right)^{1/4}. \quad (B16)$$

This expression is used in Section 9.

REFERENCES


