Vacillations Due to Wave Interference: Applications to the Atmosphere and to Annulus Experiments

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ABSTRACT

It is noted that vacillations in zonally averaged flows can arise from wave interference between coexisting waves with the same zonal wavenumber but different phase speeds. We show that the appropriate baroclinic instability problem for the annulus does yield multiple instabilities which can produce vacillations via interference. It is also shown that the interference of traveling Rossby waves with stationary forced waves can lead to vacillations whose amplitudes and periods are consistent with the observed index cycle.

1. Introduction

In studying vacillations in either rotating annulus experiments (Pfeffer *et al.*, 1980a,b) or in the atmosphere (Winston and Krueger, 1961), one frequently conceptualizes the phenomenon in terms of nonlinear baroclinic instability (Pedlosky, 1977; Boville, 1980). Briefly, the relevant models envisage an initially baroclinically unstable zonal flow where the growing baroclinic waves diminish the baroclinicity of the zonally averaged flow. With diminishing baroclinicity the baroclinic waves decrease in amplitude, allowing the baroclinicity to redevelop, restarting the whole process. To be sure, the nonlinear evolution of baroclinic instability need not always lead to vacillations, but theoretical models suggest that such vacillations can occur. In a careful attempt to simulate two-layer results experimentally, Hart (1972) found vacillations in modest agreement with nonlinear theories. However, as Boville (1980) has noted, the theoretical models predict vacillations differing in many respects from those observed in other experiments; in particular the vacillation period observed is often much shorter than predicted (Pfeffer *et al.*, 1980a).

The purpose of the present paper is to explore an alternative explanation of vacillations. In studying baroclinic instability [for the Eady problem (e.g., Charney, 1973)] one can derive the following equations for the time rate of change of the energy associated with the eddy and the zonally averaged basic states:

$$\frac{d}{dt}\int\int \frac{1}{2}[\overline{(\nabla\psi')^2} + \epsilon\overline{(\psi'_z)^2}]\rho_s dydz$$
$$= \int\int \overline{\psi'_x\psi'^{*}_y}\bar{u}_y\rho_s dydz + \int\int \epsilon\overline{\psi'_x\psi'_z}\bar{u}_z\rho_s dydz$$
$$\equiv \text{r.h.s.}, \qquad (1)$$

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$$\frac{d}{dt}\int\int \frac{1}{2}|(\nabla\bar{\psi})^2 + \epsilon\bar{\psi}_z^2|\rho_s dydz = -\text{r.h.s.}, \qquad (2)$$

where

- ψ geostrophic streamfunction
- x eastward coordinate
- y northward coordinate
- z height
- ϵ stability parameter $[=f^2/N^2]$, f being the Coriolis parameter and N the Brunt-Väisälä frequency
- $\bar{u}$ zonal velocity
- ρ_s density
- $(\bar{\ })$ $\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}(\ \ )dx.$

Eqs. (1) and (2) are merely a statement that the sum of the energy of both eddy and zonally averaged fields remains constant; i.e., eddies grow at the expense of the mean flow and vice versa. When Eqs. (1) and (2) are applied to unstable eddies the results are self-evident. However, Eqs. (1) and (2) are equally relevant to another situation where instability is not specifically at issue. Namely, if we have two waves with the same horizontal wavenumber, non-orthogonal vertical structures and different phase speeds, such waves will interfere with each other constructively and destructively with a time period dependent on the wave periods. From Eqs. (1) and (2) we see that this periodic interference must lead to a vacillation in the mean flow as well. The main purpose of this paper will be to show that this simple interference effect is a *plausible* explanation for some vacillations in both the rotating annulus and the atmosphere.

In the atmosphere, we simultaneously have forced stationary long waves as well as traveling Rossby waves with the same wavenumbers. The interferences of these two waves must lead to vacillations in the mean flow with periods equal to the Rossby wave periods. We shall consider this case in Sections 3 and 4. The situation in the annulus is more complicated. The Eady problem is usually considered appropriate to the annulus. For the Eady problem (Eady, 1949) there exists a short-wave cutoff for instability. Beyond this cutoff, there do, indeed, exist two solutions for each wavenumber whose vertical structures are non-orthogonal and whose phase speeds are different. Unfortunately, these solutions are neutral and should decay in the presence of friction. Below the cutoff wavenumber, these two solutions coalesce into a *single* unstable mode and interference cannot occur.

In Section 2 we reexamine the baroclinic instability problem for the annulus using profiles of static stability suggested by annulus measurements rather than the constant static stabilities appropriate to the Eady problem. Such changes dramatically alter the vertical distribution of horizontal gradients of potential vorticity leading to multiple instabilities at a given wavenumber and hence to the possibility of interference-induced vacillations. Indeed, a consideration of the stability properties of the more realistic basic state yields immediate insights into the nature of amplitude as well as tilted trough vacillations. It also clearly demonstrates the importance of choosing the correct linear problem before proceeding to nonlinear calculations.

In Section 3, a simple barotropic model is introduced [similar to that employed by Charney and Devore (1979)] wherein the interference of stationary waves with traveling Rossby waves of the same wavenumber can be easily studied. This, in fact, appears to be the simplest configuration in which interference-induced vacillations can be studied. From observations (Eliasen and Machenhauer, 1965, 1969), we know that such stationary waves and Rossby waves do coexist in the atmosphere, though the origin of the latter is not clear.

In Section 4, we use the data of Eliasen and Machenhauer, as well as data independently analyzed, to estimate the relative magnitudes of stationary and traveling waves, and assign to the traveling waves the frequencies calculated by Kasahara (1980). Our simple barotropic model is, of course, too simple to be accurately applicable to the atmosphere. Nonetheless, it yields vacillation amplitudes and periods encouragingly similar to those associated with observed index cycles in the atmosphere.

With respect to both vacillations in the annulus and in the atmosphere, the present study is clearly preliminary in nature. The nature of vacillations produced by wave interference is trivially evident from Eqs. (1) and (2). The purpose of this paper, as we have already stated, is merely to render plausible the operation of this mechanism in real situations.

2. Baroclinic instability in the annulus

For quasi-geostrophic flow in a Boussinesq fluid on an f-plane, conservation of potential vorticity leads to the following equation for perturbations on a basic state $\bar{u}(z)$:

$$\frac{d}{dz}\left(\epsilon\frac{d\psi}{dz}\right) + \left[\frac{-\frac{d}{dz}\left(\epsilon\frac{d\bar{u}}{dz}\right)}{\bar{u}-c} - k^2\right]\psi = 0, \quad (4)$$

with boundary conditions

$$\psi_z - \frac{\frac{d\bar{u}}{dz}}{\bar{u}-c}\psi = 0 \quad \text{at} \quad z = 0, 1, \quad (5)$$

where

$$\epsilon = \frac{f^2}{N^2},$$

$$N^2 = \frac{g}{\Theta}\frac{\partial\Theta}{\partial z},$$

and Ψ, the geostrophic streamfunction, may be written

$$\Psi = \psi(z)e^{ik(x-ct)}.$$

All lengths have been scaled by the depth H of the fluid [*viz.*, Charney (1973) and Green (1960) for development of these equations].

For convenience, we will rewrite (4) in canonical form; i.e., letting $\varphi = \sqrt{\epsilon}\psi$, Eqs. (4) and (5) become

$$\varphi_{zz} + \left[\frac{(-\epsilon_z\epsilon^{-1}\bar{u}_z - \bar{u}_{zz})}{\bar{u}-c} - \frac{k^2}{\epsilon} + \frac{1}{4}\left(\frac{\epsilon_z}{\epsilon}\right)^2 - \frac{1}{2}\frac{\epsilon_{zz}}{\epsilon}\right]\varphi = 0, \quad (6)$$

and

$$\varphi_z - \left(\frac{\epsilon_z}{2\epsilon} + \frac{\bar{u}_z}{\bar{u}-c}\right)\varphi = 0 \quad \text{at} \quad z = 0, 1, \quad (7)$$

or in terms of N^2

$$\varphi_{zz} + \left[\frac{-\left(\frac{(N^2)_z}{N^2}\bar{u}_z - \bar{u}_{zz}\right)}{\bar{u}-c} - \frac{N^2}{f^2}k^2 + \left\{\frac{(N^2)_{zz}}{2N^2} - \frac{3}{4}\left(\frac{(N^2)_z}{N^2}\right)^2\right\}\right]\varphi = 0, \quad (8)$$

and

$$\varphi_z - \left[\frac{\bar{u}_z}{\bar{u}-c} - \frac{(N^2)_z}{2N^2}\right]\varphi = 0 \quad \text{at} \quad z = 0, 1. \quad (9)$$

Now

$$\frac{(N^2)_z}{N^2}\bar{u}_z - \bar{u}_{zz} = \bar{q}_y$$

$$= y - \text{gradient of potential vorticity.} \quad (10)$$

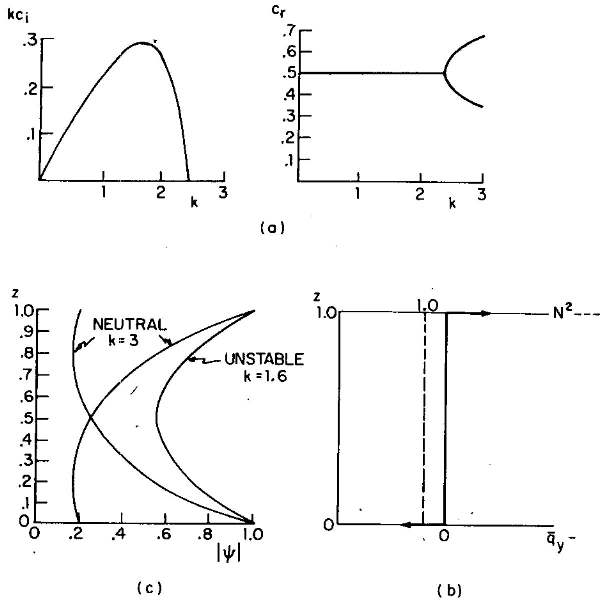

FIG. 1a. Eigenvalues of the Eady problem. Nondimensional phase speed c_r and growth rate kc_i are shown as functions of nondimensional wavenumber k.

FIG. 1b. The gradient of basic-state potential vorticity $\bar{q}_y$ and the square of the Brunt-Väisälä frequency N^2 as a function of height z for the Eady problem.

FIG. 1c. Examples of neutral (k = 3.0) and unstable k = 1.6 eigenfunctions for the Eady problem.

We also will define

$$\gamma = \frac{(N^2)_{zz}}{2N^2} - \frac{3}{4}\left(\frac{(N^2)_z}{N^2}\right)^2. \qquad (11)$$

For the Eady problem $\bar{u}_z$ and N^2 are constant. Hence $\bar{q}_y = 0$ in the interior. However, as shown in Bretherton (1966) and in Lindzen and Tung (1978), the boundary conditions (9) are equivalent to having $\bar{u}_z = 0$ at $z = 0, 1$, and having δ-function contributions to $\bar{u}_{zz}$ at the boundaries. This gives a negative contribution to $\bar{q}_y$ at $z = 0$ and a positive contribution at $z = 1$. Thus, the Eady problem's basic state involves a change in sign of $\bar{q}_y$ as required for instability (e.g., Charney, 1973). The Eady problem is peculiar insofar as its inflection "point" consists in the entire interior flow. The stability properties of the Eady problem are shown in Fig. 1a, where c_r and kc_i are shown as functions of k (c is normalized by $\bar{u}$ at z = 1 while k is normalized by $1/\sqrt{\epsilon}$). As noted in Section 1, there is only a single unstable mode at any unstable k, and hence, there is no obvious potential for interference. Also shown are the (trivial) distributions of $\bar{q}_y$ and N^2 in Fig. 1b and examples of both neutral and unstable eigenfunctions in Fig. 1c.

Now Lindzen and Tung (1978) noted that an inflection point was needed to establish a geometry for wave overreflection. The necessary geometry is such that if a fluid allows wave propagation in some region (ignoring the imaginary part of c), then a wave in this region can be overreflected (reflection coefficient > 1) by a critical level (where $c = \bar{u}$) when the critical level is separated from the propagation region by a region of exponential behavior and when there exists another region of wave propagation on the other side of the critical level. When an overreflected wave is contained by a wall or turning point it can become an unstable model provided it satisfies a quantization condition. Detailed discussions of the relation of overreflection to baroclinic instability are given in Lindzen *et al.* (1980), and Lindzen and Rosenthal (1981). What we wish to emphasize here is that increasing the number of "inflection points" (where $\bar{q}_y = 0$) multiplies the number of possible overreflecting wave regions and hence leads to the possibility of multiple instabilities.

Looking at Eq. (10) we see that when either $(N^2)_z$ or $\bar{u}_{zz}$ or both are nonzero then we have $\bar{q}_y \neq 0$. If we assume, for the moment, that $(N^2)_z = 0$, it is clear that only a fairly complicated profile in $\bar{u}$ can lead to additional inflection points and it seems unlikely that in practice such profiles will be sufficiently complicated to lead, by themselves, to multiple instabilities. On the other hand, if we hold $\bar{u}_z$ constant, simple variations in N^2 can lead to profound changes in $\bar{q}_y$—including additional inflection points and multiple instabilities. The situation of variable N^2 is complicated by the term γ [Eq. (11)] which essentially modifies the effective value of k insofar as it affects baroclinic instability.

As an example of the effect of variable N^2 on baroclinic instability, we consider the basic state

$$\bar{u}_z = \text{constant}, \qquad (12)$$

$$N^2(z) = 0.3 + 0.7 \exp\left[-\left(\frac{z - 0.5}{0.1}\right)^2\right]. \qquad (13)$$

N^2, as well as $\bar{T}(z)$, are shown in Fig. 2; also shown is $\bar{q}_y(z)$. We see that a modest change in $\bar{T}(z)$ such as to concentrate N^2 in the middle of the fluid, radically alters $\bar{q}_y(z)$—replacing the single "inflection point" of the Eady problem with three "inflection points." Eqs. (8) and (9), with $\bar{u}_z$ and $N^2(z)$ given by Eqs. (12) and (13), are solved numerically by an algorithm described in the Appendix. The results of the stability analysis are shown in Fig. 3. Fig. 3a shows c_r and kc_i as a function of k (c is normalized by $\bar{u}$ at $z = 1$ while k is normalized by $1/\sqrt{\epsilon}|_{z=0}$). We see that for $k > 1.5$, we now have two unstable modes for each k, while for $k < 1.5$ there is only a single unstable mode. Note also that there are separate growth rate peaks associated with both the multiple mode and the single mode regime, the latter being the larger of the two. Fig. 3b shows eigenfunctions for various choices of k. For $k < 1.5$ the eigenfunction resembles that for the Eady problem. For $k > 1.5$,

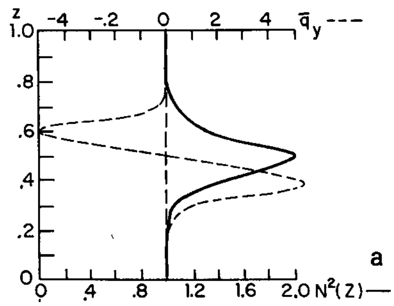

FIG. 2a. The gradient of basic-state potential vorticity $\bar{q}_y$ and the square of the Brunt-Väisälä frequency N^2 as a function of height z for the example (13).

where we have two unstable solutions, their degree of overlap increases as $k \rightarrow 1.5$.

It is tempting to suggest that these two regimes may be identified with the amplitude vacillation and the tilted trough vacillation regimes in annulus experiments (Pfeffer *et al.*, 1974). As one moves through the regime diagram for an annulus one finds that at high thermal Rossby numbers [$\mathrm{Ro_T} = g\alpha H\Delta T/\Omega^2 r^2$, where g is the gravitational acceleration, α the coefficient of volume expansion, H the fluid depth, ΔT the imposed temperature contrast, Ω the rate of rotation of annulus, and $r = (b - a)$ where b is the outside radius and a the inside radius of the annulus], the annulus is not wide enough to contain low-k modes and one indeed does find amplitude vacillations for which interference is now a possible explanation. At lower values of $\mathrm{Ro_T}$ the annulus is wide enough for low-k modes and tilted trough vacillations are observed.

From recent results of annulus experiments (privately communicated by Pfeffer) similar to those presented in Pfeffer *et al.* (1980b), we find the description of $\bar{u}$ still insufficiently detailed. However, substantial data are available for $\bar{T}(z)$. These data do show a concentration of N^2 in the interior of the fluid. However, the peak of N^2 is found below the middle of the fluid. The following is a fair fit to the observed N^2:

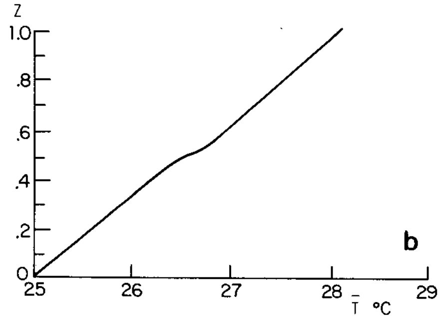

FIG. 2b. The temperature profile $\bar{T}(z)$ which modifies the Eady problem results in the values of $\bar{q}_y$ and N^2 shown in Fig. 2a.

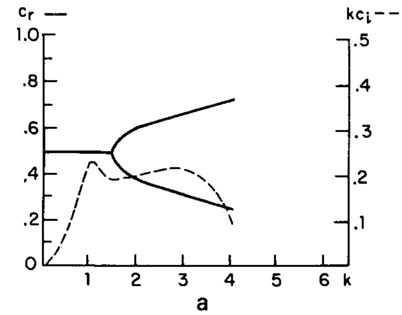

FIG. 3a. Eigenvalues of the stability problem in Fig. 2a. Nondimensional phase speed c_r and growth rate kc_i are shown as functions of nondimensional wavenumber k.

$$N^2 = 0.3 + 0.7\left\{\frac{1}{2}\left[\tanh\frac{(z - 0.15)}{0.1} - \tanh\frac{(z - 0.7)}{0.2}\right]\right\}. \quad (14)$$

We have investigated the stability properties of a basic state described by (12) and (14). Fig. 4 shows $N^2(z)$ as well as $\bar{q}_y(z)$. Once again we have three "inflection points." In Fig. 5 we show stability results (c, here, is normalized by $\bar{u}$ at $z = 1$ and k is normalized by $1/\sqrt{\epsilon_{\min}}$). Double instabilities are also found here for larger values of k, while a single Eady-like instability is found at small k. Eigenfunctions are shown in Fig. 6. Obvious differences can be seen between the results in Fig. 5 and those in Fig. 3. However, these differences can be significantly modified by modest alterations of $\bar{u}(z)$. Detailed analyses of the results do not seem warranted until more se-

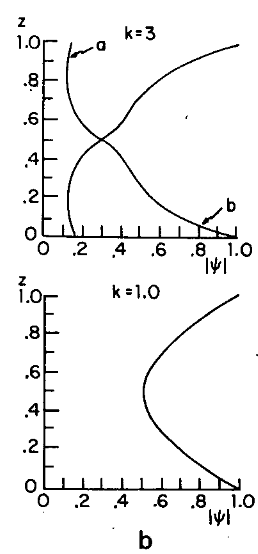

FIG. 3b. Eigenfunctions for the stability problem in Fig. 2a. Examples are shown where one wave exists at a given k(k = 1.0) and where two modes exist (k = 3.0).

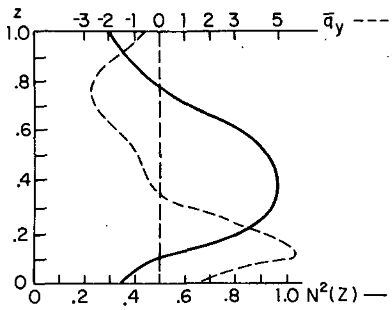

FIG. 4. As in Fig. 2a except for N^2 given by Eq. (14).

cure and detailed data are available for the basic state.

Nevertheless, it should be noted that if two unstable modes exist with wavenumber k, and real phase speeds c_{r_1} and c_{r_2}, then amplitude vacillations will occur with frequency $k(c_{r_1} - c_{r_2})$. (Recall that k is scaled by $1/\sqrt{\epsilon_{\min}}$ and c_r by $\bar{u}$ at $z = 1$.) The rather long period for some observed amplitude vacillations [$\sim$130 rotation periods according to Pfeffer *et al* (1980b)] thus favors small $\bar{u}_z$ in the basic state and values of k near the transition to single-mode instability. Such values of k also are associated with maximum overlap of the two unstable modes. Indeed for larger values of k, the degree of overlap between the two unstable eigenfunctions may be too small to permit much interference at all.

It should be noted that the wave interference mechanism explains the wide variation in observed vacillation periods (Pfeffer *et al.*, 1980a,b), a result not easily accommodated by nonlinear theories (Boville, 1980), which also predict weak and irregular cycles for realistic parameter values in variance with the extremely regular and strong amplitude vacillation seen in the annulus experiments.

In a recent private communication, Pfeffer and Buzyna have made available additional data for the series A and B experiments described in Pfeffer *et al.* (1980a). The vacillation is represented as the superposition of two traveling waves with the same wavenumber M, but differing in frequency ω_1, ω_2:

$$F_M(x, t) = C\cos(Mx - \omega_1 t) + D\cos(Mx - \omega_2 t). \quad (15)$$

They determine values for C, D, ω_1 and ω_2 and find expression (15) gives a striking fit to the data. In addition, phase speeds ω_1/M, ω_2/M are typical of $\frac{1}{3}H$ and $\frac{2}{3}H$ depths in the annulus and eigenfunctions are very like those in Fig. 3b, strongly suggesting that in these cases at least the vacillation is due to wave interference.

It appears that our interference hypothesis may be most effectively tested experimentally. For our theory to be rigorously convincing one would have to show that our mechanism persists in a fully nonlinear analysis. Such an analysis would have to have sufficient vertical resolution in order to handle our multiple instability mechanism. Such a study is not contemplated at the moment. It is conceivable [as shown in some recent nonlinear analyses such as Hart (1978)] that one instability might, in fact, destroy the other. However, studies displaying such behavior are usually dealing with unstable modes of different wavenumbers occupying the same physical space in the fluid. In our case we are dealing with instabilities having identical horizontal wavenumbers but occupying and drawing their energy from different parts of the fluid. It seems unlikely that nonlinear effects will allow one of the multiple modes to "absorb" the other.

What we do anticipate, as we have already mentioned, is that as $\mathrm{Ro_T}$ decreases so as to allow the Eady-type mode in addition to the multiple instabilities, that the Eady-type mode may very well eliminate the other instabilities and produce a tilted trough (as opposed to an amplitude) vacillation.

3. Vacillations due to interference of Rossby waves with stationary waves

Although the essence of the vacillation mechanism we are describing is given by Eqs. (1) and (2), detailed calculations of vacillations can be complicated. An exception is the case of interfering forced stationary and free traveling Rossby waves in a barotropic fluid. The study of this case illustrates the mechanism we are proposing. Moreover, the results of this section form the basis for our discussion of the atmospheric index cycle in Section 4.

The equation describing this situation is the quasi-geostrophic potential vorticity equation on a β-plane [essentially the same equation used by Charney and Devore (1979)]:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_G\cdot\nabla\right)(\nabla^2\psi + \beta y) + \mathbf{v}_G\cdot\frac{f_0\nabla h}{H} = 0, \quad (16)$$

where $\mathbf{v}_G$ is the geostrophic wind, ψ the geostrophic streamfunction, β the rate of variation of Coriolis

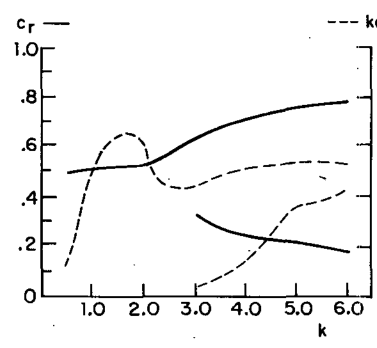

FIG. 5. As in Fig. 3a except for N^2 given by Eq. (14).

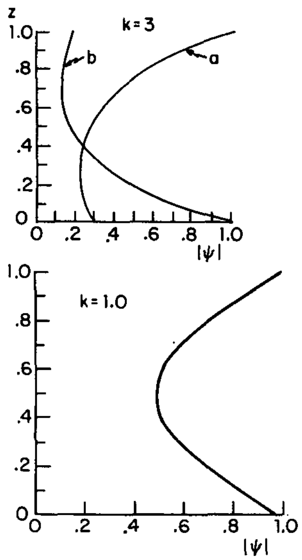

FIG. 6. As in Fig. 3b except for N^2 given by Eq. (14).

parameter, and h height of bottom topography. Our boundary condition is

$$\psi_x = 0 \quad \text{at} \quad y = 0, l. \tag{17}$$

We next let

$$\left.\begin{aligned} \psi &= -u_0 y + \psi' \\ h &= h' \end{aligned}\right\}. \tag{18}$$

Primed quantities are taken to be small so that we may expand (16) in amplitude.

To first order we have the linearized wave equation

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right)(\nabla^2\psi') + \beta \frac{\partial \psi'}{\partial x} + \frac{u_0 f_0}{H}\frac{\partial h'}{\partial x} = 0. \tag{19}$$

The last term in (19) represents stationary forcing due to flow over topography. Let

$$h' = \mathrm{Re}(\hat{h} \sin my e^{ikx}), \tag{20}$$

where $m = \pi/l$. The solution to (19) can be written

$$\psi' = \psi' \text{ forced} + \psi' \text{ free}, \tag{21}$$

where

$$\psi' \text{ forced} = \mathrm{Re}[\hat{\psi} \sin my e^{ikx}], \tag{22}$$

$$\psi' \text{ free} = \mathrm{Re}[\tilde{\psi} \sin my e^{ik(x-ct)}]. \tag{23}$$

Substituting (22) into (19) yields

$$\hat{\psi} = \frac{1}{H} \frac{u_0 \hat{h} f_0}{u_0(k^2 + m^2) - \beta} \equiv \mathcal{A}. \tag{24}$$

Substituting (23) into the homogeneous part of (19) [i.e., leaving out $(u_0/H)(\partial h'/\partial x)$] we have

$$c = u_0 - \frac{\beta}{k^2 + m^2}. \tag{25}$$

The amplitude of the free solution, of course, is arbitrary. We will let $\tilde{\psi} = \alpha\hat{\psi} = \alpha\mathcal{A}$, and for convenience we will let α be real. Then

$$\psi' = \mathcal{A}\{\cos kx + \alpha \cos[k(x - ct)]\} \sin my. \tag{26}$$

Eq. (26) may be rewritten

$$\begin{aligned} \psi' &= \mathcal{A}\{(1 + \alpha \cos kct) \cos kx \\ &\quad + \alpha \sin kct \sin kx\} \sin my \qquad (27) \\ &= \mathcal{A}A(t) \cos(kx + \varphi(t)) \sin my, \end{aligned}$$

where

$$A(t) = (1 + \alpha^2 + 2\alpha \cos kct)^{1/2}, \tag{28}$$

$$\varphi(t) = -\tan^{-1}\left(\frac{\alpha \sin kct}{1 + \alpha \cos kct}\right). \tag{29}$$

The time variation of $A(t)$ results from the interference of the forced and free waves. Note, also, that the phase progression given by (29) is nonuniform.

Time variation of A is obviously akin to amplitude vacillation—the degree of vacillation depending, of course, on the relative magnitudes of the forced and free waves; the period of vacillation is given simply by the period of the free wave. The variation of eddy kinetic energy (in our barotropic model there is no potential energy) is easily calculated using

averaged Eddy Kinetic Energy

$$= \mathrm{EKE} = l^{-1}\int_0^l dy\, \tfrac{1}{2}\overline{(u'^2 + v'^2)}, \tag{30}$$

where

$$u' = -\psi'_y = -\mathcal{A}mA(t) \cos[kx + \varphi(t)] \cos my, \tag{31}$$

$$v' = \psi'_x = -\mathcal{A}kA(t) \sin[kx + \varphi(t)] \sin my. \tag{32}$$

Using (31), (32) and (28), Eq. (30) becomes

$$\mathrm{EKE} = \tfrac{1}{8}\mathcal{A}^2(m^2 + k^2)(1 + \alpha^2 + 2\alpha \cos kct). \tag{33}$$

The time-dependent part of (33) is

$$\tfrac{1}{4}\mathcal{A}^2(m^2 + k^2)\alpha \cos kct$$

or using (25) and (24)

$$\frac{u_0}{4}\frac{f_0}{H}\frac{\mathcal{A}\hat{h}\alpha}{c} \cos kct. \tag{34}$$

We know from Eqs. (1) and (2) that (34) is balanced by an identical (except for sign) variation in the energy of the zonally averaged flow. The explicit calculation of the variation of zonal flow, however, is of interest.

The vacillation in zonally averaged flow due to wave interference is calculated by considering (16) at second order, i.e.,

$$\frac{\partial}{\partial t}\nabla^2\psi'' + u_0\frac{\partial}{\partial x}\nabla^2\psi'' - \psi'_y\frac{\partial}{\partial x}\nabla^2\psi' + \psi'_x\frac{\partial}{\partial y}\nabla^2\psi' + \beta\psi''_x - \left(\psi'_y\frac{\partial h'}{\partial x} - \psi'_x\frac{\partial h'}{\partial y}\right)\frac{f_0}{H} = 0. \tag{35}$$

Averaging (35) with respect to x, we get

$$\frac{\partial}{\partial t}\frac{d^2}{dy^2}\overline{\psi''} - \overline{\psi'_y \frac{\partial}{\partial x}\nabla^2\psi'} + \overline{\psi'_x \frac{\partial}{\partial y}\nabla^2\psi'}$$
A B

$$- \left(\overline{\psi'_y \frac{\partial h'}{\partial x}} - \overline{\psi'_x \frac{\partial h'}{\partial y}}\right)\frac{f_0}{H} = 0. \quad (36)$$
C D

It is easily shown that terms A and B in (36) cancel. For terms C and D, we get

$$-\overline{\psi'_y h'_x} + \overline{\psi'_x h'_y} = \tfrac{1}{2}\mathcal{A}\hat{h}\alpha mk \sin 2my \sin kct, \quad (37)$$

and (36) becomes

$$-\frac{\partial}{\partial t}\frac{d^2}{dy^2}\overline{\psi''} = f_0 H^{-1}(\tfrac{1}{2}\mathcal{A}\hat{h}\alpha mk \sin 2my \sin kct). \quad (38)$$

Integrating with respect to t, Eq. (38) becomes

$$\overline{\psi''_{yy}} = \frac{1}{2}\frac{f_0}{H}\frac{\mathcal{A}\hat{h}\alpha m}{c}\sin 2my \cos kct, \quad (39)$$

and integrating with respect to y Eq. (39) becomes

$$\overline{\psi'_y} = -\bar{u}''$$
$$= -\frac{1}{2}\frac{\mathcal{A}\hat{h}\alpha m}{c}\frac{f_0}{H}\cos kct \times \left(\frac{1}{2m}\cos 2my - \text{const}\right). \quad (40)$$

To determine the constant in (40) we require that

$$\bar{u}'' = 0 \quad \text{at} \quad y = 0 \quad (41)$$

(because no momentum is transported to the lateral boundaries) and (40) becomes

$$\bar{u}'' = \frac{1}{4}\frac{\mathcal{A}\hat{h}\alpha}{c}\frac{f_0}{H}\cos kct(\cos 2my - 1). \quad (42)$$

We may now directly evaluate the kinetic energy of the zonally averaged zonal flow:

$$\frac{1}{2}\frac{1}{l}\int_0^l \bar{u}^2 dy$$
$$= \frac{1}{2l}\int_0^l (u_0^2 + 2u_0\bar{u}'' + \bar{u}''^2)dy$$
$$= \frac{1}{2}u_0^2 + \frac{u_0}{l}\int_0^l\left(-\frac{1}{4}\frac{\mathcal{A}\hat{h}\alpha}{c}\cos kct\right)dy + O(h^4)$$
$$= \frac{1}{2}u_0^2 - \frac{u_0}{4}\frac{f_0}{H}\frac{\mathcal{A}\hat{h}\alpha}{c}\cos kct + O(h^4). \quad [43]$$

We see that the time-dependent part (43) does indeed balance (34).

Finally, we note that if we have a set of forced modes of the form

$$\hat{\psi}(m, n)\cos mk[x - \delta_{h(m,n)}]\sin\frac{n\pi}{y_T}y, \quad (44)$$

then (42) is generalized to

$$\bar{u}'' = \frac{f_0}{4H}\sum_{m,n}\frac{h(m, n)}{c(m, n)}\tilde{\psi}(m, n) \times \cos\{mk[c(m, n)t + \delta_{\tilde{\psi}(m,n)} - \delta_{h(m,n)}]\} \times \left(\cos\frac{2n\pi}{y_T}y - 1\right). \quad (45)$$

It is worth noting from (45) that the effectiveness of a pair of forced and free waves in generating vacillations in $\bar{u}$ is inversely proportional to $c(m, n)$.

4. Estimates for vacillations in the atmosphere

Eliasen and Machenhauer (1965) have analyzed the Northern Hemisphere flow field for the period 1 December 1956–28 February 1957. They decomposed the flow into spherical harmonic components and studied the fluctuation with time of the components.

They represented the Northern Hemisphere flow field as a sum of spherical harmonics, of the form

$$\psi_{m,q} = (\alpha_{m,q}\cos m\lambda + \beta_{m,q}\sin m\lambda)P_{m,q}(\varphi)$$
$$= A_{m,q}\cos m(\lambda - \delta_{m,q})P_{m,q}(\varphi). \quad (46)$$

Here λ is the zonal coordinate and φ the meridional coordinate. The $P_{m,q}$ are the normalized associated Legendre functions of the first kind. Eliasen and Machenhauer restricted the $P_{m,q}$ to be odd about the equator ($q - m$ odd). The $\psi_{m,q}$ were nondimensionalized such that dimensional $\mathbf{v}$ equals $10^{-4}\Omega R^2\mathbf{k} \times \nabla\psi$. R denotes the radius of the earth and ∇ is the spherical gradient operator.

Eliasen and Machenhauer reported the amplitude $A^*_{m,q}$ and phase $\delta^*_{m,q}$ of each of the waves $\psi_{m,q}$ $1 \leqslant m \leqslant 4$, $1 \leqslant q - m \leqslant 7$ in the 90-day *mean* flow. This is taken to be the stationary component of the wave. They also gave the time-mean amplitude $\hat{A}_{m,q}$ of each wave as

$$A^*_{(m,q)} = [(\tilde{\alpha}_{m,q})^2 + (\tilde{\beta}_{m,q})^2]^{1/2}, \quad (47a)$$

$$\hat{A}_{(m,q)} = (\alpha^2_{m,q} + \beta^2_{m,q})^{1/2}, \quad (47b)$$

$$(\ ) = T^{-1}\int_0^T (\ )dt$$

indicates the time average. (47c)

Finally, for waves (1, 2), (1, 4), (2, 3), (2, 5), (3, 4) and (3, 6) they reported the apparent phase speed

TABLE 1. Data reported by Eliasen and Machenhauer.

(m, q)	$\hat{A}_{(m,q)}$	$A^*_{(m,q)}$	$\delta^*_{(m,q)}$ (deg longitude)	$c_{(m,q)}$ (deg longitude per day)	Period τ Observed (days)	Period τ (Kasahara) (days)
(1, 2)	9.5	8.8	330	−70	5.1	4.85
(1, 4)	9.6	3.9	55	−20	18	18.39
(2, 3)	6.2	4.1	52	−40	4.5	3.84
(2, 5)	20.1	17.9	33	−12	15	14.23
(3, 4)	7.3	5.0	91	−20	6	4.28
(3, 6)	14.4	12.3	99	−8	15	13.65

of the traveling wave component. Calculations in this paper will be restricted to these six waves. Table 1 contains the information they reported.

$\delta^*_{(m,q)}$ is in degrees longitude while $c_{(m,q)}$ is in degrees longitude per day. τ is period in days. The values in Table 1 are in good agreement with Kasahara's theoretical results for zonal wind fields representative of the December–February time period.

If the various waves were purely stationary, we would have $\hat{A}_{(m,q)} = A^*_{(m,q)}$. The difference between $\hat{A}_{(m,q)}$ and $A^*_{(m,q)}$ indicates the relative importance of transient components. If one assumes that the transient component consists only of a traveling Rossby wave of amplitude $r_{(m,q)}A^*_{(m,q)}$, then $r_{(m,q)}$ satisfies

$$\hat{A}_{(m,q)} = \pi^{-1}A^*_{(m,q)} \int_0^\pi (1 + 2r_{m,q}\cos t' + r^2_{(m,q)})^{1/2}dt' = A^*_{m,q}I(r_{(m,q)}). \quad (48)$$

Since $\hat{A}_{(m,q)}$ and $A^*_{(m,q)}$ are known the above is an implicit expression for $r_{(m,q)}$. The $r_{(m,q)}$ and $A_{t(m,q)} = r_{(m,q)}A^*_{(m,q)}$ for the six waves under consideration were calculated numerically. The results are given in Table 2. To be sure, the transient components are unlikely to consist solely of traveling Rossby waves; thus the results in Table 2 are likely to overestimate the amplitudes of the traveling waves. Nevertheless, we will use these values to estimate vacillation amplitudes and periods, using the theoretical formalism of Section 3. To do this we must transpose the spherical harmonic results of Eliasen and Machenhauer onto a β-plane. We take a β-plane centered at 45°N, extending from the equator to the pole. The zonal eigenfunctions, $\cos m(\lambda - \delta)$, are replaced with $\cos m\sqrt{2}a^{-1}(x - \delta)$, where a is the earth's radius, while the normalized meridional eigenfunctions $P_{m,q}$ are replaced by $(-1)^{n-1}2/\sqrt{\pi}\sin 2n\varphi$ (also normalized), where $2n = q - m + 1$ and $y = a\varphi$.

In terms of Section 3 the above results lead to amplitudes and phases of stationary waves and amplitudes of Rossby waves shown in Table 3. Also shown are β-plane values of $c(m, n)$ which are very close to spherical results. Eliasen and Machenhauer's (1964) analysis leads to a choice for u_0 of 7.38 m s^{-1}. For this u_0, we determine the values of $h(m, n)$ needed to produce the observed stationary waves. The results shown in Table 4 are not unreasonable. Finally, we evaluate the vacillation amplitude $\bar{u}''$ using Eq. (45) of Section 3. The results for each mode are shown in Table 5. Note that the total vacillation can involve fluctuations of ±1.5 m s^{-1} in $\bar{u}$ (compared to u_0 = 7.38 m s^{-1}) and that the second meridional eigenmodes [with periods O(18 days)] dominate the vacillation [mainly because they are associated with small values of $c(m, n)$ but also because they are strongly forced]. The period is indeed close to what one expects for the index cycle, and the period is somewhat irregular because there are several contributing modes with somewhat different periods. The amplitude of the vacillation involves changes in kinetic energy of ±40% which are substantially larger than those found in observations, but as we have already mentioned, we have probably overestimated the amplitude of traveling Rossby waves. One of the authors (R. S. Lindzen) is currently analyzing various data sets in order to get a better estimate of these amplitudes. Preliminary results do not indicate that present results are more than a factor or two too large.

5. Concluding remarks

This paper has noted that vacillations in zonally averaged fields must arise when waves of the same zonal wavenumber (but different phase speeds) interfere with each other. For the atmosphere, preliminary estimates show that interference between long stationary waves and traveling Rossby waves can account for both the period and amplitude of the index cycle. While traveling Rossby waves have been observed, our theory does not explain how they are generated.

TABLE 2. Nondimensional amplitudes A_t of the traveling waves together with the ratios $r_{(m,q)} = A_{t(m,q)}/A^*_{(m,q)}$.

(m, q)	$r_{(m,q)}$	$A_{t(m,q)}$
(1, 2)	0.56	4.9
(1, 4)	2.35	9.2
(2, 3)	1.31	5.4
(2, 5)	0.69	12.4
(3, 4)	1.25	6.3
(3, 6)	0.81	10.0

TABLE 3. Eliasen and Machenhauer data dimensionalized and transferred to the beta plane.

(m, n)	$A_{i(m,n)}$ $\times 10^{-5}$ ($m^2 s^{-1}$)	$A^*_{(m,n)}$ $\times 10^{-5}$ ($m^2 s^{-1}$)	$\delta^*_{(m,n)}$ $\times 10^{-7}$ (m)	$c_{(m,n)}$ (deg longitude per day)
(1, 1)	16.3	29.3	2.59	−63.7
(1, 2)	−30.6	−13.0	0.43	−18.2
(2, 1)	18.0	13.7	0.41	−36.4
(2, 2)	−41.3	−59.6	0.26	−10.9
(3, 1)	21.0	16.7	0.72	−18.2
(3, 2)	−33.2	−41.0	0.78	−7.3

For the annulus, we have shown that the results of the Eady problem are inappropriate, and when vertical variations in static stability are allowed, instabilities can occur in pairs (for each zonal wavenumber) thus allowing the interference mechanism to operate.

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APPENDIX

Method of Solving the Eigenvalue Problem

The continuous eigenvalue problem defined by Eq. (8) with associated boundary condition (9) can be approximated by a finite matrix eigenvalue problem as follows:

Eqs. (8) and (9) are rewritten as

$$(\bar{u} - c)\varphi_{zz} + [\alpha + (\bar{u} - c)\beta]\varphi = 0, \qquad (8a)$$

$$(\bar{u} - c)\varphi_z + [\gamma + (\bar{u} - c)\delta]\varphi = 0 \quad z = 0, 1, \qquad (9a)$$

where

$$\alpha = \frac{(N^2)_z}{N^2}\bar{u}_z - \bar{u}_{zz},$$

$$\beta = \frac{-N^2}{f^2}k^2 + \left\{\frac{(N^2)_{zz}}{2N^2} - \frac{3}{4}\left(\frac{(N^2)_z}{N^2}\right)^2\right\},$$

TABLE 4. Amplitudes of assumed forcing.

(m, n)	$h_{(m,n)}$ (m)
(1, 1)	436
(1, 2)	165
(2, 1)	189
(2, 2)	694
(3, 1)	200
(3, 2)	404

TABLE 5. Vacillation amplitudes $\bar{u}''_{(m,n)}$.

(m, n)	$\bar{u}''_{(m,n)}$ ($m s^{-1}$)
(1, 1)	0.038
(1, 2)	0.095
(2, 1)	0.032
(2, 2)	0.901
(3, 1)	0.079
(3, 2)	0.630

$$\gamma = -\bar{u}_z,$$

$$\delta = \frac{N_z^2}{2N^2}.$$

The eigenvalue c is removed to the right-hand side to yield

$$\bar{u}\varphi_{zz} + (\alpha + \bar{u}\beta)\varphi = c(\varphi_{zz} + \beta\varphi), \qquad (8b)$$

$$\bar{u}\varphi_z + (\gamma + \bar{u}\delta)\varphi = c(\varphi_z + \delta\varphi) \quad z = 0, 1. \qquad (9b)$$

Second-order finite differences are used to discretize (8) on the $N + 2$ points, $N = 0, 1, 2, \ldots, N + 1$ where $z^{(i)} = (i - 1)\Delta z$, $\Delta z = 1/(N - 2)$ and $\varphi^{(i)} = \varphi^{(i)}_{(z)}$:

$$\varphi^{(i)}_{zz} = \frac{\varphi^{(i+1)} - 2\varphi^{(i)} + \varphi^{(i-1)}}{(\Delta z)^2},$$

$$\varphi^{i}_{z} = \frac{\varphi^{(i+1)} - \varphi^{(i-1)}}{2\Delta z}.$$

The boundary conditions (9b) are discretized in the same manner and used to eliminate $\varphi^{(0)}$ and $\varphi^{(N+1)}$ in the finite-difference approximation to (8). The result is a N-dimensional matrix eigenvalue problem:

$$\mathbf{M}\varphi = c\mathbf{D}\varphi,$$

$$[\mathbf{D}^{-1}\mathbf{M}]\varphi = c\varphi.$$

The eigenvalues and eigenvectors are extracted using the highly efficient QR algorithm (Wilkinson and Reinock 1971). Accuracy of the solution verified by comparison between eigenvalues for resolutions corresponding to $N = 50$ and $N = 100$. Note that only two or four normal mode eigenvalues are found depending on whether there is one growing mode (and its associated decaying mode) or two. The other $(N - 2)$ or $(N - 4)$ are spurious.

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