On the Relation of Wave Behavior to Source Strength and Distribution in a Propagating Medium

RICHARD S. LINDZEN[1]
University of Oslo, Norway

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Analyses of the "26-month" oscillation often assume that wave amplitude and drive amplitude are directly related [e.g., Probert Jones (1964); Reed (1965)]. The purpose of this note is to indicate that this relation is, in general, valid only under two conditions: 1) when the medium does not propagate the disturbance, or 2)

[1] Now at the National Center for Atmospheric Research, Boulder, Colo.

when the thickness of the source region is small compared to the wavelength of the disturbance. Although the current analyses of these points are by no means absolutely conclusive, it does appear that 1) the stratosphere does propagate long period disturbances (Lindzen, 1966), and 2) if the "26-month" oscillation is due to ultraviolet fluctuations, the source region will in fact be quite thick (Lindzen, 1965) and not restricted to the top of the ozone layer. Since neither of the conditions for the above assumption to be correct appears to be satisfied, we must turn to the consideration of waves in a medium capable of propagating waves in only one direction [by analogy with Lindzen (1966)] with a distributed source. The simplest mathematical example of such a situation is given by

$$\frac{\partial u}{dt}-c\frac{\partial u}{\partial x}=F(x)e^{i\omega t}, \tag{1}$$

where u is some arbitrary field, c is a constant, x is the space dimension and t represents time. The right hand side of (1) represents a distributed source, harmonic in time with period $2\pi/\omega$. In the absence of a source, the medium propagates waves in the direction of $-x$ with a constant phase speed c. If we restrict ourselves to sources such that

$$F(x)\rightarrow 0 \quad \text{as} \quad x\rightarrow +\infty,$$

and to u's such that

$$u=u(x)e^{i\omega t},$$

where $u(x)\rightarrow 0$ as $x\rightarrow +\infty$, then the solution to (1) is

$$u(x)=\frac{1}{c}e^{ikx}\int_x^\infty F(x')e^{-ikx'}dx', \tag{2}$$

where $k\equiv\omega/c$. Note that for a medium which does not propagate, $c=0$ and $u(x)=-(i/\omega)F(x)$. In this case the amplitude of the disturbance follows the amplitude of the drive.

When $c\neq 0$, the situation is somewhat more complicated. The simplest example would concern a source with a constant amplitude in a layer of finite thickness and zero amplitude elsewhere. It is a simple matter to show that for such a situation, the response amplitude in the drive region oscillates between zero and some maximum, with the zeroes being separated by $2\pi/k$ (i.e., a wavelength). In this case, it is perfectly clear that the amplitude of u is not a simple indicator of the local drive amplitude. Moreover, the phase speed that would be observed between the zeroes in the amplitude would be $2c$ not c. These two features, i.e., that 1) observed amplitude is not simply related to the drive distribution, and 2) observed phase speed in the drive region will not be that inherent to the medium, are also to be seen in the following two examples involving more complicated drive distributions.

a) Let

$$\left.\begin{aligned}
&F=0 && \text{for} \quad x>x_0\\
&F=\lambda(x_0-x) && \text{for} \quad x_0>x>x_1\\
&F=\lambda\frac{(x_0-x_1)}{(x_1-x_2)}(x-x_2) && \text{for} \quad x_1>x>x_2\\
&F=0 && \text{for} \quad x_2>x
\end{aligned}\right\}. \tag{3}$$

Then

$$\left.\begin{aligned}
&u=0 \quad \text{for} \quad x_i>x_0\\
&u=\frac{\lambda}{ck^2}[\{1-\cos k(x_0-x)\}+i\{\sin k(x_0-x)-k(x_0-x)\}] \quad \text{for} \quad x_0>x>x_1\\
&u=\frac{\lambda}{ck^2}\left\{\left[\left(1+\frac{(x_0-x_1)}{(x_1-x_2)}-\cos k(x_0-x_1)\right)\cos k(x_1-x)+\sin k(x_0-x_1)\sin k(x_1-x)-\frac{(x_0-x_1)}{(x_1-x_2)}\right]\right.\\
&\qquad\left.+i\left[\left(\cos k(x_0-x_1)-1-\frac{(x_0-x_1)}{(x_1-x_2)}\right)\sin k(x_1-x)+\sin k(x_0-x_1)\cos k(x_1-x)-k(x_0-x_1)\frac{(x-x_2)}{(x_1-x_2)}\right]\right\}\\
&\qquad\qquad \text{for} \quad x_1>x>x_2\\
&u=u(x_2)e^{ik(x-x_2)} \quad \text{for} \quad x<x_2
\end{aligned}\right\}. \tag{4}$$

Although (4) is too complicated for a detailed discussion, some of its important features are shown in Fig. 1 where F, $|u|$, and the phase of u are shown for the special case where $(x_0-x_1)/(x_1-x_2)=\frac{1}{4}$ and $k(x_1-x_2)=2\pi$. Note that the maximum of $|u|$ occurs at a level below the maximum of F, and that the apparent phase speed in the source region is greater than c.

b) As a last example we briefly consider

$$F(x)=\lambda e^{-\sigma x^2}. \tag{5}$$

Eq. (2) then becomes

$$u=-\frac{\lambda}{c}\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{\sigma}}e^{-(k^2/4\sigma)}e^{i(\omega t+kx)}\operatorname{erfc}\left[\sqrt{\sigma}\left(x+i\frac{k}{2\sigma}\right)\right], \quad (6)$$

where erfc refers to the complementary error function. In general the complementary error function has a rather complicated behavior when the real and imaginary parts of its argument are comparable in magnitude. The behavior, however, is not significantly different from that which obtains with (3) when $(x_0-x_1)/(x_1-x_2)=1$. We will here remark on only two points.

When

$$\sqrt{\sigma}x\rightarrow-\infty, \quad \operatorname{erfc}\left[\sqrt{\sigma}\left(x+\frac{ik}{2\sigma}\right)\right]\rightarrow 2,$$

and

$$u\rightarrow-\frac{\lambda}{c}\frac{1}{\sqrt{\sigma}}e^{-(k^2/4\sigma)}e^{i(\omega t+kx)}. \quad (7)$$

Eq. (7) displays an important feature of driven waves, i.e., the amplitude of the disturbance far down from the source depends on the width of the source (i.e., on σ). In particular, (7) has a maximum when $\sigma=\frac{1}{2}k^2$. For a thicker source (i.e., for $\sigma<\frac{1}{2}k^2$), the amplitude far from the source is actually reduced, even when the amplitude of the source distribution remains contant.

Finally we consider the matter of a "thin" source region, i.e., $\sigma^2\gg k^2$. In this case (6) becomes

$$u\cong-\frac{\lambda}{c}\frac{1}{\sqrt{\sigma}}\frac{\sqrt{\pi}}{2}e^{i(\omega t+kx)}\operatorname{erfc}(\sqrt{\sigma}x). \quad (8)$$

Since erfc $(\sqrt{\sigma}x)$ rises rapidly from something close to zero for $\sqrt{\sigma}x>1$ to approximately 2 for $\sqrt{\sigma}x<-1$, the disturbance reaches its maximum amplitude near the bottom of the source region and since the source region is "thin", we can associate this level approximately with the source level itself.

The above examples are too simple to be directly applicable to the problem of the "26-month" wind. If the wind does result from ultraviolet fluctuations then the source distribution will differ from any of those in the above calculation (Lindzen, 1965). Moreover, the dispersion properties of the stratosphere are likely to be quite complicated. In particular, c probably has an important dependence on x, i.e., altitude (Lindzen,

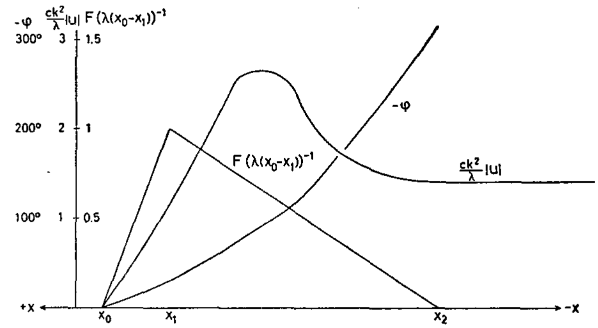

FIG. 1. Source intensity F, disturbance amplitude $|u|$ and disturbance phase φ, as functions of x. Case a) with $\frac{(x_0-x_1)}{(x_1-x_2)}=\frac{1}{4}$ and $k(x_1-x_2)=2\pi$. See text for details.

1966). Nevertheless, in the above examples we do see what happens when a disturbance at a particular level is activated both directly and by energy propagated from above. These two sources of activation interfere with each other. The results are that the presence of a distributed source affects the apparent phase speed in the source region, and that the ability of the medium to propagate the disturbance affects the local relation between disturbance and source amplitude. These results must hold equally for more complicated situations and lead to the pessimistic conclusion that if one knows nothing about either the dispersive properties of the medium or the source distribution, then inferences from data about either in isolation from the other are almost impossible.

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