

## A WKB Asymptotic Analysis of Baroclinic Instability

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### ABSTRACT

A modest extension of conventional WKB methods of solving second-order differential equations asymptotically is developed and applied to the problem of baroclinic instability. Accurate results are obtained for the Charney problem using simple expressions. More important, the present expressions can be used immediately for a wide variety of basic velocity profiles.

### 1. Introduction

The Charney problem in baroclinic instability (the instability of a constant shear profile in a stratified, semi-infinite atmosphere), while of clear relevance to meteorology, has solutions (involving confluent hypergeometric functions) which are rather complicated and difficult to approach intuitively (see Charney, 1947, 1973; Pedlosky, 1979, for treatments of this problem). Indeed, Charney (1947) did not get so far as to actually calculate growth rates. The complexity of the rather idealized Charney problem hardly invites further generalization. Moreover, in most graduate courses in dynamic meteorology, the Charney problem tends to be ignored with emphasis given instead to the far less realistic, far more problematic, but also far simpler, two-layer (Phillips, 1954) and Eady (1949) models.

The recent work of Lindzen and Tung (1978) and Lindzen *et al.* (1980) suggests that all problems in baroclinic and barotropic instability [including the Charney problem, but excluding symmetric overturning (Charney, 1973)] involve the interaction of Rossby waves with critical surfaces (where mean flow equals phase speed) in the presence of specific combinations of regions of wave propagation and trapping. To be sure, in the general problem critical surfaces are neither purely horizontal nor purely vertical. While this complicates the mathematics, it does not fundamentally alter the basic physics of the instability. Numerical results for the more general problem clearly show the relevance of the simpler Charney problem where the critical surface is horizontal (Simmons and Hoskins, 1976; Gall, 1976). Given the wave interpretation of baroclinic instability, one might expect that WKB methods (Bender and Orszag, 1978) which specifically isolate wavelike and exponential behavior might prove useful. In this paper, a modest extension of WKB analy-

sis is applied to the Charney problem. The resulting solutions are quantitatively accurate and readily extended to more general problems.

### 2. Equations

The equations for the Charney problem have been amply discussed in the literature. The discussion in Lindzen *et al.* (1980) most nearly parallels the present approach. Restricting ourselves to quasi-geostrophic perturbations on a basic state where the zonal flow is a function of height  $\bar{u}(z)$ , the Brunt-Väisälä frequency  $N$  and the scale height  $H$  are constants and furthermore, using a  $\beta$ -plane geometry, we obtain the equation for the perturbation streamfunction  $\Psi$

$$\psi_{zz} + \left[ \frac{q_y(z)}{\bar{u} - c} - \frac{k^2}{\epsilon} - \frac{1}{4H^2} \right] \psi = 0, \quad (1)$$

where solutions of the form

$$\Psi = \psi(z)e^{z/(2H)}e^{ik(x-ct)} \quad (2)$$

have been assumed and where

$$\begin{aligned} q_y(z) &= \frac{\beta}{\epsilon} + \frac{1}{H} \frac{d\bar{u}}{dz} - \frac{d^2\bar{u}}{dz^2} \\ &= \frac{\partial}{\partial y} (\text{pseudo-potential vorticity}), \end{aligned} \quad (3)$$

where

- $H$  scale height for basic density (assumed constant) [=  $RT_0/g$ ]
- $z$  height
- $\epsilon$   $f^2/N^2$
- $f$  Coriolis parameter
- $\beta$   $df/dy$
- $x$  eastward distance.

Requiring the vertical velocity to vanish at the ground implies

$$\psi_z + \frac{1}{2H} \psi - \frac{d\bar{u}/dz}{\bar{u} - c} \psi = 0 \quad \text{at } z = 0. \quad (4)$$

For the upper boundary condition it will prove adequate to assume solutions decay as  $z \rightarrow \infty$ .

Restricting ourselves more specifically to the Charney problem we will take

$$\bar{u}(z) = u_0 + mz. \quad (5)$$

Then, under the following nondimensionalization:

$$\bar{u} \equiv (\bar{u} - u_0)/(mH), \quad (5a)$$

$$\bar{c} \equiv (c - u_0)/(mH), \quad (5b)$$

$$\bar{z} \equiv z/H, \quad (5c)$$

$$\alpha^2 \equiv (k^2 H^2)/\epsilon, \quad (5d)$$

Eqs. (1) and (4) become

$$\psi_{\bar{z}\bar{z}} + \left( \frac{r+1}{\bar{z} - \bar{c}} - \delta^2 \right) \psi = 0, \quad (6)$$

$$\psi_{\bar{z}} + \frac{1}{2} \psi + \frac{1}{\bar{c}} \psi = 0, \quad \text{at } \bar{z} = 0, \quad (7)$$

where

$$r = \beta H / \epsilon m, \quad (8)$$

$$\delta^2 = \alpha^2 + 1/4. \quad (9)$$

It proves useful for what follows to replace  $\bar{z}$  by  $\zeta$  where

$$\zeta \equiv \bar{z} - \bar{c}, \quad (10)$$

in which case (6) and (7) become

$$\psi_{\zeta\zeta} + \left( \frac{r+1}{\zeta} - \delta^2 \right) \psi = 0, \quad (11)$$

$$\psi_{\zeta} + \frac{1}{2} \psi + \frac{1}{\bar{c}} \psi = 0, \quad \text{at } \zeta = -\bar{c}. \quad (12)$$

To be sure, since  $\bar{c}$  may be complex, we now have the awkwardness of a complex vertical coordinate. However, for purposes of discussion, it will prove convenient to deal with  $\zeta$  as though it were real. Such an approach is physically justified by the success of the wave overreflection approach to baroclinic instability (Lindzen *et al.*, 1980). More important, our mathematical results remain valid in the relevant portions of the complex plane. The question of validity is discussed further at the end of Section 3, but it should be noted that the fact that representations for real  $\zeta$  carry over to unstable solutions, without crossing any Stokes lines, shows that instabilities are simply analytic continuations of overreflected waves.

### 3. Basic analysis

Eq. (11) is a traditional wave equation where the quantity in brackets may be viewed as a measure of the index of refraction. Let

$$m^2 \equiv \left( \frac{r+1}{\zeta} - \delta^2 \right). \quad (13)$$

The  $\zeta$  domain is characterized by three special points:  $\zeta = \zeta_0 = -\bar{c}$ , the ground;  $\zeta = \zeta_s = 0$ , a regular singularity; and  $\zeta = \zeta_t = (r+1)/\delta^2$ , a turning point.  $m^2$  is negative above  $\zeta_t$  (indicating exponential behavior), positive between  $\zeta_s$  and  $\zeta_t$  (indicating oscillatory or wavelike behavior) and negative between  $\zeta_0$  and  $\zeta_s$ . Note that these properties of  $m^2$  are strictly correct only for real values of  $\zeta$ . When the imaginary part of  $c$  is nonzero, real  $\zeta$  corresponds to complex  $z$ .

We will first concentrate on the immediate neighborhood of  $\zeta = \zeta_s = 0$ . The quantity  $m^2$  is dominated by the singular part. As  $\zeta \rightarrow 0$ , the solutions of (11) will approach the solutions of

$$\bar{\psi}_{\zeta\zeta} + [(r+1)/\zeta] \bar{\psi} = 0. \quad (14)$$

Eq. (14) is a form of Bessel's equation, and its solutions may be found in most textbooks on differential equations (Boyce and DiPrima, 1977; Kamke, 1948):

$$\bar{\psi} = 2[(r+1)\zeta]^{1/2} \times (AH_1^{(1)}\{2[(r+1)\zeta]^{1/2}\} + BH_1^{(2)}\{2[(r+1)\zeta]^{1/2}\}) \quad (15)$$

for  $\zeta > 0$ , where  $H_j^{(k)}$  is a Hankel function of the  $k$ th kind and  $j$ th order, and  $A$  and  $B$  are constants.

In the neighborhood of  $\zeta_s$ ,

$$\psi \approx \bar{\psi}. \quad (16)$$

For  $\zeta > \zeta_s$ , but small enough so that  $(r+1)/\zeta \gg \delta^2$ ,

$$H_1^{(1)} \sim (2/\{\pi[2\sqrt{(r+1)\zeta}]\})^{1/2} \times \exp\{i[2\sqrt{(r+1)\zeta} - 3\pi/4]\}, \quad (17)$$

$$H_1^{(2)} \sim (2/\{\pi[2\sqrt{(r+1)\zeta}]\})^{1/2} \times \exp\{-i[2\sqrt{(r+1)\zeta} - 3\pi/4]\}, \quad (18)$$

$$\psi \sim \left( \frac{2}{\pi} \right)^{1/2} [2\sqrt{(r+1)\zeta}]^{1/2} \times \{A \exp[i(2\sqrt{(r+1)\zeta} - 3\pi/4)] + B \exp[-i(2\sqrt{(r+1)\zeta} - 3\pi/4)]\}. \quad (19)$$

$H_1^{(1)}$  is readily shown to correspond to an upward propagating internal Rossby wave, while  $H_1^{(2)}$  corresponds to a downward propagating wave.

Next we note that for  $\zeta_s < \zeta < \zeta_t$ , Eq. (11) has the WKB solution

$$\psi \approx m^{-1/2}(A'e^{i\zeta} + B'e^{-i\zeta}), \quad (20)$$

where

$$Z = \int_0^\zeta m d\zeta \tag{21}$$

and  $A'$  and  $B'$  are constants.

Now, for  $(r + 1)/\zeta \gg \delta^2$ ,

$$Z \cong 2[(r + 1)\zeta]^{1/2} \tag{22}$$

and

$$m^{-1/2} = \left( \frac{r + 1}{\zeta} - \delta^2 \right)^{-1/4} \approx (\zeta^{1/4})/[(r + 1)^{1/4}]. \tag{23}$$

Thus (19) and (20) are essentially identical in this region. This leads to the trivial construction of the following solution which behaves like both (15) and (20) in the appropriate regions:

$$\psi \cong \left( \frac{r + 1}{\zeta} - \delta^2 \right)^{-1/4} \times \sqrt{Z} [AH_1^{(1)}(Z) + BH_1^{(2)}(Z)]. \tag{24}$$

Away from  $\zeta_s$ , (24) has the asymptotic form

$$\psi \approx \left[ \frac{r + 1}{\zeta} - \delta^2 \right]^{-1/4} \sqrt{2/\pi} \times [A \exp(iZ - 3\pi i/4) + B \exp(-iZ + 3\pi i/4)] \tag{25}$$

which, in effect, is identical to (20).

Our next task is to determine  $A$  and  $B$  (to within a multiplicative constant). To do this we make use of our upper boundary condition: namely, that  $\psi$  decays as  $\zeta \rightarrow \infty$ . This requires that (24) (appropriate for  $\zeta < \zeta_t$ ) be matched to that solution for  $\zeta > \zeta_t$  which decays with increasing  $\zeta$ . This problem of matching WKB solutions across an ordinary turning point is standard in the literature (Bender and Orszag, 1978), and involves the use of Airy functions in a manner analogous to the use of Hankel (or Bessel) functions in the present paper. We shall omit the readily obtainable details here, merely noting that the solution for  $\zeta < \zeta_t$  corresponding to exponential decay for  $\zeta > \zeta_t$  is given by

$$\psi \propto \cos\left(Z_T - Z - \frac{\pi}{4}\right), \tag{26}$$

where

$$Z_T = \int_0^{\zeta_t} m d\zeta = \pi(r + 1)/2\delta. \tag{27}$$

[Recall that  $\delta = (\alpha^2 + 1/4)^{1/2}$ .] The integral in Eq. (27) is integrated by elementary methods [or lacking the patience for such methods, one may look up the solution in Dwight (1957)]. Eq. (26) may be rewritten

$$\psi \propto \exp[i(Z_T - Z - 1/4\pi)] + \exp[-i(Z_T - Z - 1/4\pi)] \tag{28}$$

and comparison of (28) with (25) yields

$$\left. \begin{aligned} A &\propto \exp[i(\pi - Z_T)] \\ B &\propto \exp[-i(\pi - Z_T)] \end{aligned} \right\}, \tag{29}$$

and, finally (24) becomes

$$\psi \approx C \left( \frac{r + 1}{\zeta} - \delta^2 \right)^{-1/4} \sqrt{Z} \{ \exp[i(\pi - Z_T)] H_1^{(1)}(Z) + \exp[-i(\pi - Z_T)] H_1^{(2)}(Z) \}, \tag{30}$$

where  $C$  is a constant.

The dispersion relation for the evaluation of the problem's stability properties [i.e.,  $\tilde{c}(\alpha)$ ] comes from the lower boundary condition (12). In order to obtain this relation, (30) must be continued to negative values of  $\zeta$ . The continuation is simple and described in adequate detail in the literature (Watson, 1966; Abramowitz and Stegun, 1965; Morse and Feshbach, 1961). The exact continuation, however, depends on whether negative  $\zeta$  is approached by going above or below  $\zeta = 0$  in the complex plane. The latter case is appropriate for both growing instabilities, and for neutral solutions in the presence of slight damping. Thus for negative  $\zeta$ , we take  $\zeta = |\zeta|e^{-i\pi}$ , when evaluating (30). This yields

$$\psi \approx C' \left( \frac{r + 1}{x} + \delta^2 \right)^{-1/4} \sqrt{X} \times (\exp[i(\pi - Z_T)] \{ -(2/\pi)[-K_1(X) + \pi I_1(X)] \} + \exp[-i(\pi - Z_T)] \{ -(2/\pi)K_1(X) \}) \tag{31}$$

or

$$\psi \approx C'' [(r + 1)/x + \delta^2]^{-1/4} \sqrt{X} [-2 \sin(\pi - Z_T) K_1(X) + \pi \exp[i(\pi - Z_T)] I_1(X)], \tag{32}$$

where  $C'$ ,  $C''$  are constants,  $x = -\zeta$ ,

$$X = \int_0^x [\delta^2 + (r + 1)/x]^{1/2} dx, \tag{33}$$

and  $I_1$  and  $K_1$  are Bessel functions of imaginary argument. Clearly,  $(H_1^{(1)}, H_1^{(2)})$  and  $(I_1, K_1)$  are the Bessel equation counterparts of oscillatory and exponential functions, respectively.  $I_1$  is regular, while  $K_1$  is singular; it is evident from (32) that whenever  $\pi - Z_T = n\pi$  ( $n = 0, \pm 1, \pm 2, \dots$ ),  $\psi$  is regular. Not surprisingly, it is exactly at these points that the Charney problem has neutral solutions. This will be discussed further later in this paper.

One could, at this point, consider the asymptotic behavior of (32) [in a manner analogous to (24)]. However, since we will be interested in small values of  $\tilde{c}$ , it will prove more accurate to deal with (32) as it stands.

For the lower boundary condition we need

$$\begin{aligned} \psi_x = -\psi_\zeta = -\psi_z \approx C'' \{ & \{ [\delta^2 + (r+1)/X]^{1/4} / (2\sqrt{X}) + (r+1)\sqrt{X} [\delta^2 + (r+1)/X]^{-5/4} / (4X^2) \} \\ & \times [-2 \sin(\pi - Z_T) K_1(X) + \pi \exp[i(\pi - Z_T)] I_1(x)] + \sqrt{X} [\delta^2 + (r+1)/x]^{1/4} [-2 \sin(\pi - Z_T) K_1'(X) \\ & + \pi \exp[i(\pi - Z_T)] I_1'(X)] \}. \quad (34) \end{aligned}$$

From (12), (32) and (34) we obtain our equation for  $\bar{c}(\alpha)$ :

$$\begin{aligned} \left( \frac{1}{2} + \frac{1}{\bar{c}} - \left\{ \frac{1}{2X(\bar{c})} [\delta^2 + (r+1)/\bar{c}]^{1/2} + \frac{(r+1)}{4\bar{c}^2} [\delta^2 + (r+1)/\bar{c}]^{-1} \right\} \right) \times \{ & [-2 \sin(\pi - Z_T)] \pi^{-1} \\ & \times \exp[-i(\pi - Z_T)] K_1[X(\bar{c})] + I_1[X(\bar{c})] \} - [\delta^2 + (r+1)/\bar{c}]^{1/2} \{ [-2 \sin(\pi - Z_T)] \pi^{-1} \\ & \times \exp[-i(\pi - Z_T)] K_1'[X(\bar{c})] + I_1'[X(\bar{c})] \} = 0, \quad (35) \end{aligned}$$

where

$$\left. \begin{aligned} K_1'(X) &= -\frac{1}{2} [K_0(X) + K_2(X)] \\ I_1'(X) &= \frac{1}{2} [I_0(X) + I_2(X)] \end{aligned} \right\}, \quad (36)$$

and

$$X(\bar{c}) = \bar{c} [(r+1)/\bar{c} + \delta^2]^{1/2} + [(r+1)/\delta] \times \ln \{ (\bar{c}/(r+1))^{1/2} \{ \delta + [(r+1)/\bar{c} + \delta^2]^{1/2} \} \} \quad (37)$$

(viz., integral 195.04 in Dwight, 1957).

Eq. (35), while hardly trivial, is readily solved by standard methods for complex root determination. More important is the obvious fact that the extension of (35) to profiles of  $\bar{u}(z)$  other than that of the Charney problem requires only a change in the expressions for  $Z_T$  and  $X(\bar{c})$ , and in the quantity  $[\delta^2 + (r+1)/\bar{c}]$  (providing that the problem still has a meaningful counterpart to  $z_t$ , the turning point). The present mathematics clearly reflect the physical identity of all baroclinic wave instabilities (as described in Section 1). In problems where, for small  $\alpha$ 's, there is no  $z_t$ , a similar solution is still obtained using the so-called "full-wave" approach of radio physics (Budden, 1961). A similar situation arises when we have a rigid lid below the effective turning point for a given  $\alpha$ . The solution in this case is given in the Appendix to this paper. In principle, a problem might have several turning points but, in practice, such problems are not geophysically common.

The explicit solution of (35) is dealt with in the following sections.

*Note concerning validity of asymptotics:* In order to avoid unnecessarily dwelling on the details of the asymptotic procedures, we have focused on the analysis for real  $\zeta$ . The analysis, however, is valid for complex  $c$  (and complex  $\zeta$ ). For  $\text{Re}(\zeta) > 0$ , (17) is valid for  $-\pi < \arg \zeta < 2\pi$  and (18) is valid for  $-2\pi < \arg \zeta < \pi$ . Real values of  $z$  always lie within these limits in the  $\zeta$ -plane. Similarly, below the turning point (26) is valid for  $|\arg(Z - Z_T)| < 2\pi/3$ . Again, for  $\text{Re}(\zeta) < \text{Re}(\zeta_t)$ , real  $z$  always lies within these limits in the  $\zeta$ -plane. The limits of the sectors of validity are known as Stokes lines and are discussed in Bender and Orszag (1978). The explicit Stokes lines cited above are given in Abramowitz and Stegun (1965).

It should be noted that the evaluation of the integral in (27) is unaffected when  $c$  is complex since the integration path can be taken along real  $\zeta$ .

#### 4. Stability results

The solution of Eq. (35) is facilitated by the fact that both  $I_1(X)$  and  $K_1(X)$  have rapidly convergent series representations

$$I_1(X) = \frac{X}{2} + \frac{X^3}{16} + \frac{X^5}{384} + \frac{X^7}{18,432} + \dots, \quad (38a)$$

$$\begin{aligned} K_1(X) &= \frac{1}{X} + X \left\{ \frac{1}{2} \ln \frac{X}{2} + 0.038608 \right\} + X^3 \left( 0.0625 \ln \frac{X}{2} - 0.042049 \right) \\ &+ X^5 \left( 0.0026042 \ln \frac{X}{2} - 0.00283711 \right) \\ &+ X^7 \left( 0.00005425 \ln \frac{X}{2} - 0.000074930 \right) + \dots \end{aligned} \quad (38b)$$

Given (27), (36), (37) and (38), everything in Eq. (35) is specified. Eq. (35) was solved using the secant method described in Lindzen and Rosenthal (1976). In Figs. 1a and 1b we show (for  $r = 1$ )  $\bar{c}_r$  vs.  $\alpha$

and  $\bar{c}_i$  vs.  $\alpha$ , respectively. Results are shown for an accurate numerical integration of (11), and for the solution of (35) wherein (i) nine terms were retained

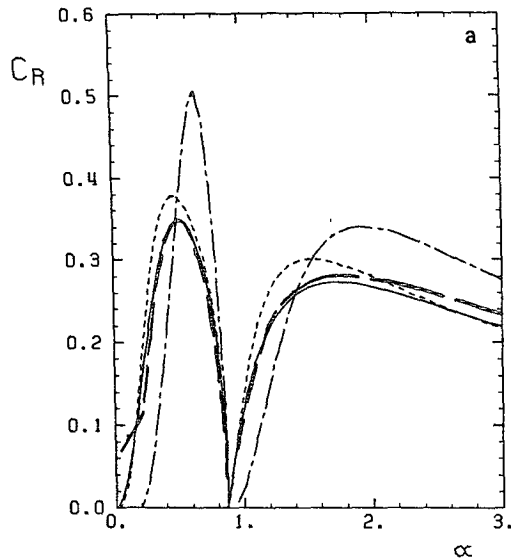


FIG. 1a. Nondimensional phase speed  $\bar{c}_r$  as a function of non-dimensional horizontal wavenumber  $\alpha$ . The line with long dashes shows results from numerical solutions (obtained with a lid at  $\bar{z} = 16$ ); the solid line shows WKB results where nine terms were retained in Bessel function expansions; the shorter dashes show results where two terms were retained in Bessel function expansions; and the combination with long and short dashes shows results where two terms were retained in the expansion of  $I_1$ , but only one term in the expansion of  $K_1$ . All results in this figure are for  $r = 1$ .

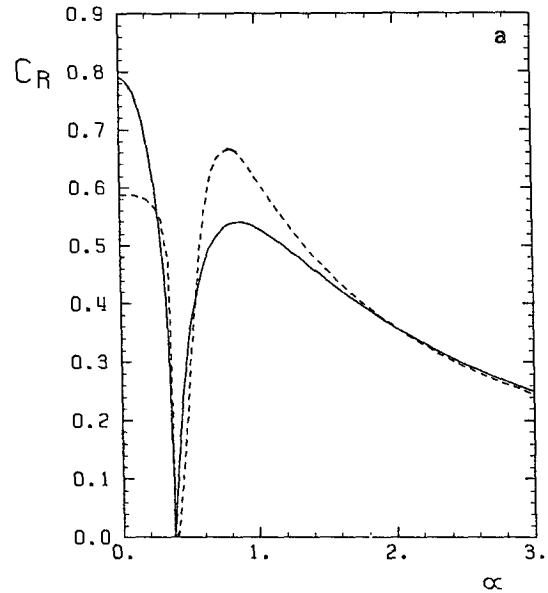


FIG. 2a. As in Fig. 1a except that  $r = 0.25$ , and results are shown only for the nine-term Bessel function expansions (solid line) and the two term expansions (dashed line).

in each of (38a) and (38b); (ii) two terms were retained in each of (38a) and (38b); and (iii) two terms were retained in (38a), but only one term was retained in (38b).

It is evident that (35) is a rather good approxima-

tion to the exact numerical solution, and that the retention of only two terms in (38a) and (38b) is adequate for many purposes. Even case (iii) represents a fair approximation. Although not shown in Figs. 1a and 1b, the retention of only one term in (38) leads to nonsensical results. This feature will be explained in the following section.

Finally, (35) proves to be a comparably good approximation at all values of  $r$ . However, the accuracy of the two term approximations to the Bessel

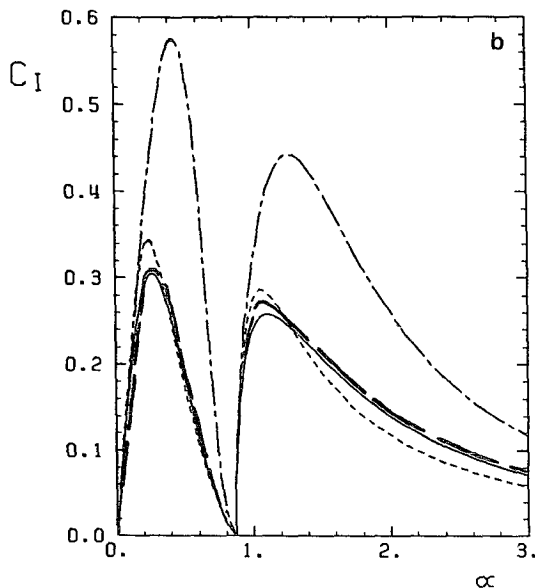


FIG. 1b. As in Fig. 1a except for  $\bar{c}_i$  as a function of  $\alpha$ .

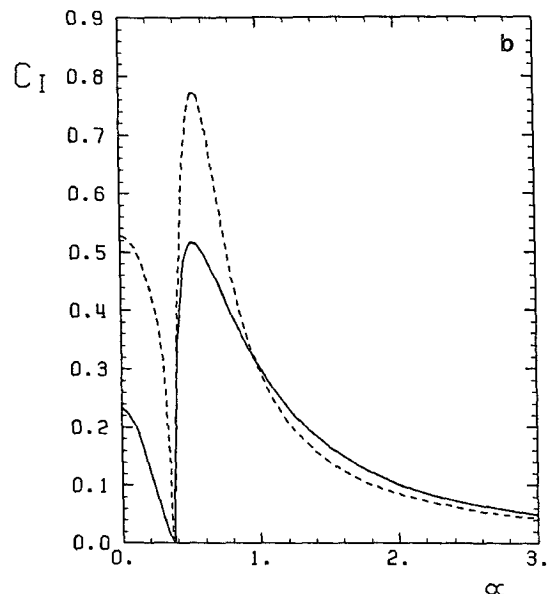


FIG. 2b. As in Fig. 2a except for  $\bar{c}_i$  as a function of  $\alpha$ .

functions diminishes somewhat at small values of  $r$ . This is seen in Figs. 2a, 2b, 3a and 3b where results for cases (i) and (ii) above are compared at  $r = 0.25$  and  $r = 2.0$ .

5. Expansions in  $\bar{c}$

Since in Figs. 1-3,  $|\bar{c}| \ll 0(1)$ , one might expect that (35) could be further simplified by expanding everything in terms of powers of  $\bar{c}$ . Thus, for example,

$$[\delta^2 + (r + 1)\bar{c}]^{1/2} = [(r + 1)\bar{c}]^{1/2} [1 + (\delta^2\bar{c})/(r + 1)]^{1/2} = [(r + 1)\bar{c}]^{1/2} \{1 + \delta^2\bar{c}/[2(r + 1)] + \dots\}, \quad (39)$$

$$[\delta^2 + (r + 1)\bar{c}]^{-1} = \bar{c}^{-1} [1 - \delta^2\bar{c}/(r + 1) + \dots]/(r + 1), \quad (40)$$

$$X = 2[(r + 1)\bar{c}]^{1/2} \left\{ 1 + \frac{1}{6} \frac{\delta^2}{(r + 1)^2} [(r + 1)\bar{c}] - \frac{1}{40} \frac{\delta^4}{(r + 1)^4} [(r + 1)\bar{c}]^2 + \dots \right\}. \quad (41)$$

Substituting (39), (40) and (41) into (32) and (34) using (38a) and (38b) is a tedious though straightforward task. The results are

$$\begin{aligned} \psi(\bar{c}) \approx C'' & \left( \pi \exp[i(\pi - Z_T)] \sqrt{\frac{2}{r + 1}} \left[ \boxed{(r + 1)\bar{c}} + \frac{1}{2}(r + 1)^2\bar{c}^2 + (\frac{1}{12} + \frac{1}{6}\Delta^2 + \frac{1}{15}\Delta^4)(r + 1)^3\bar{c}^3 + \dots \right] \right. \\ & - 2 \sin(\pi - Z_T) \sqrt{\frac{2}{r + 1}} \left\{ \frac{1}{2} + (r + 1)\bar{c} \ln\sqrt{(r + 1)\bar{c}} + (0.077216 - \frac{1}{6}\Delta^2)(r + 1)\bar{c} \right. \\ & + \frac{1}{2}(r + 1)^2\bar{c}^2 \ln\sqrt{(r + 1)\bar{c}} + (-0.336392 + \frac{1}{6}\Delta^2 + \frac{1}{10}\Delta^4)(r + 1)^2\bar{c}^2 \\ & \left. \left. + (r + 1)^3\bar{c}^3 [\ln\sqrt{(r + 1)\bar{c}}] (\frac{1}{12} + \frac{1}{6}\Delta^2 + \frac{1}{15}\Delta^4) + \dots \right\} \right) \quad (42) \end{aligned}$$

$$\begin{aligned} \psi_x \approx C'' & \left( \pi \{ \exp[i(\pi - Z_T)] \} \sqrt{2(r + 1)} \left[ \boxed{1} + (r + 1)\bar{c} + (r + 1)^2\bar{c}^2(\frac{1}{4} + \frac{1}{2}\Delta^2 + \frac{1}{5}\Delta^4) + \dots \right] \right. \\ & - 2[\sin(\pi - Z_T)] \sqrt{2(r + 1)} \left\{ \ln\sqrt{(r + 1)\bar{c}} + (0.5772158 - \frac{1}{6}\Delta^2) + (r + 1)\bar{c} \ln\sqrt{(r + 1)\bar{c}} \right. \\ & \left. \left. + (r + 1)\bar{c}(-0.422784 + \frac{1}{3}\Delta^2 + \frac{1}{5}\Delta^4) + (r + 1)^2\bar{c}^2 [\ln\sqrt{(r + 1)\bar{c}}] \left( \frac{1}{4} + \frac{\Delta^2}{2} + \frac{\Delta^4}{5} \right) + \dots \right\} \right), \quad (43) \end{aligned}$$

where  $\Delta = \delta/(r + 1)$ .

The dispersion relation is obtained by substituting (42) and (43) into (12) (N.B.  $\psi_c = -\psi_x$ ):

$$\begin{aligned} & \left[ \left( \frac{1}{2} + \frac{1}{\bar{c}} \right) \psi - \psi_x \right] [(r + 1)2]^{1/2} \frac{\bar{c} \exp[-i(\pi - Z_T)]}{\pi C''} \\ & = \underbrace{\left\{ -\frac{1}{2}r(r + 1)\bar{c}^2 + \bar{c}^3 [(-\frac{1}{6} - \frac{1}{3}\Delta^2 - \frac{2}{15}\Delta^4)(r + 1)^3 + \frac{1}{4}(r + 1)^2] + \dots \right\}}_{1a} \\ & \quad - \frac{2}{\pi} \sin(\pi - Z_T) \exp[-i(\pi - Z_T)] \left\{ \underbrace{\frac{1}{2}}_{1b} - \underbrace{\frac{1}{2}(r + 1)\bar{c}}_{2b} + \underbrace{\bar{c}^2 [\ln\sqrt{(r + 1)\bar{c}}] [-\frac{1}{2}r(r + 1)]}_{3b} \right\} \\ & \quad + \underbrace{\bar{c}^2 \left[ \frac{1}{2}(r + 1)(0.077216 - \frac{1}{6}\Delta^2) + (r + 1)^2(0.086392 - \frac{1}{6}\Delta^2 - \frac{1}{10}\Delta^4) \right]}_{4b} \\ & \quad + \underbrace{\bar{c}^3 [\ln\sqrt{(r + 1)\bar{c}}] (r + 1)^2 \left[ \frac{1}{4} + (r + 1)(-\frac{1}{6} - \frac{1}{3}\Delta^2 - \frac{2}{15}\Delta^4) \right] + \dots}_{5b} = 0. \quad (44) \end{aligned}$$

Note that contributions from the boxed terms in (42) and (43) to  $(1/c)\psi$  and  $\psi_x$ , respectively, cancel. This explains why the retention of only one term in (38)

leads to nonsensical results. The second term in (38) contributes vitally to the leading term remaining after cancellation.

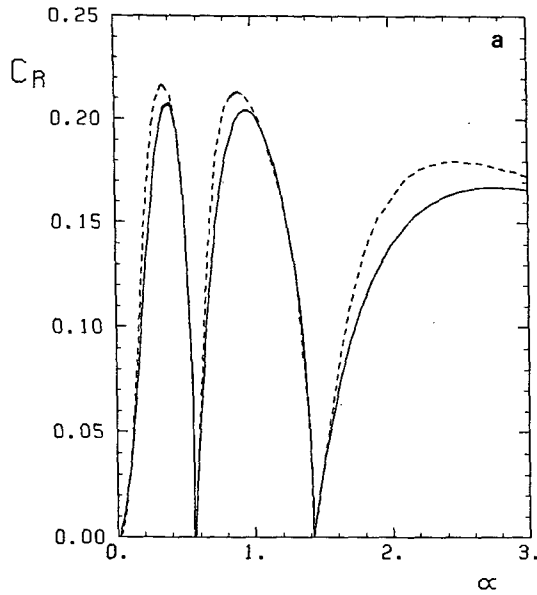


FIG. 3a. As in Fig. 2a except that  $r = 2.0$ .

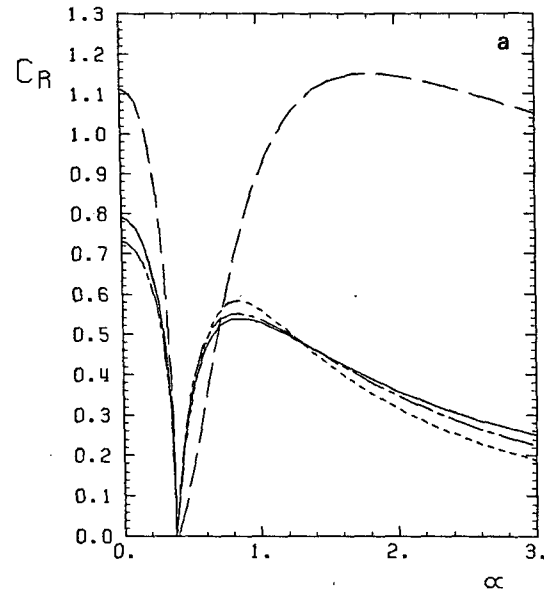


FIG. 4a.  $\bar{c}_r$  as a function of  $\alpha$  for cases (i) shown by (—), (ii) shown by (---), (iii) shown by (- - -) and (iv) shown by (—) described in the text.  $r = 0.25$ .

In Figs. 4a, 4b, 5a, 5b, 6a and 6b we show  $\bar{c}_r$  and  $\bar{c}_i$  vs  $\alpha$  for  $r = 0.25, 1$  and  $2$  for the following truncations of (44): (i) retention of terms  $1a$  and  $1b$ ; (ii) retention of terms  $1a, 2a, 1b$  and  $2b$ ; (iii) retention of all terms shown in (44); (iv) the results from Section 4 based on nine-term expansions.

We see that the terms shown in (44) represent a generally excellent approximation. More remarkably, the two-term approximation [case (ii)], is equally good for all values of  $r$ . The one term approximation wherein (44) is approximated by the

very simple expression

$$\bar{c}^2 \approx -\frac{2}{\pi} \{ \sin(\pi - Z_T) \} \{ \exp[-i(\pi - Z_T)] \} \div [r(r + 1)] \quad (45)$$

is, unfortunately, not very accurate, but it does give the qualitative behavior of the stability curves (especially the neutral points), and reasonable order of magnitude values—especially at larger values of  $r$ .

It should be noted that the dependence of the

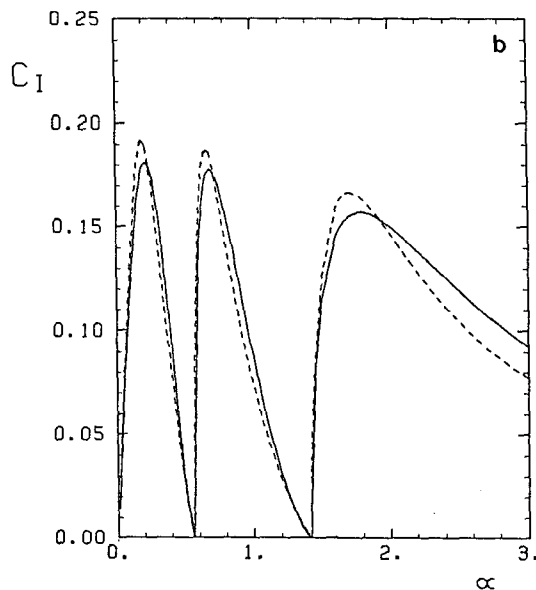


FIG. 3b. As in Fig. 2b except that  $r = 2.0$ .

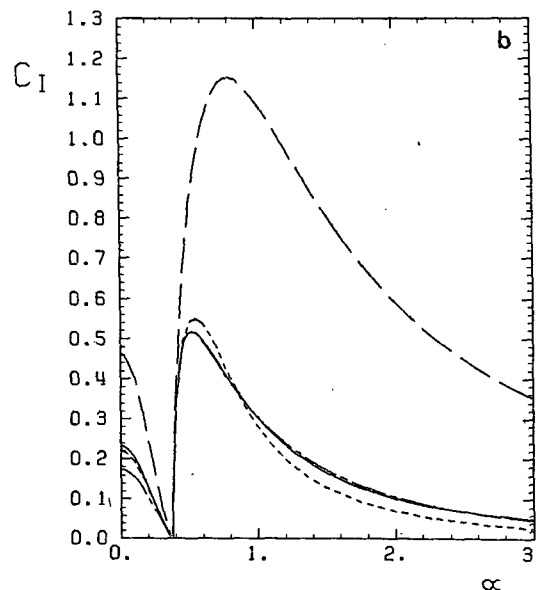


FIG. 4b. As in Fig. 4a except for  $\bar{c}_i$  as a function of  $\alpha$ .

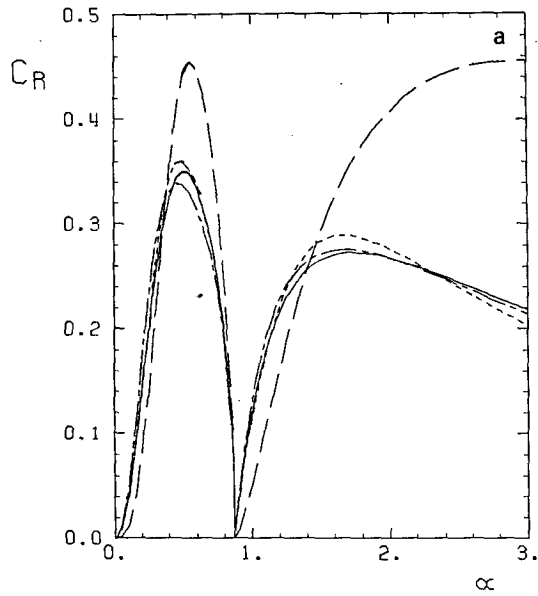


FIG. 5a. As in Fig. 4a except for  $r = 1$ .

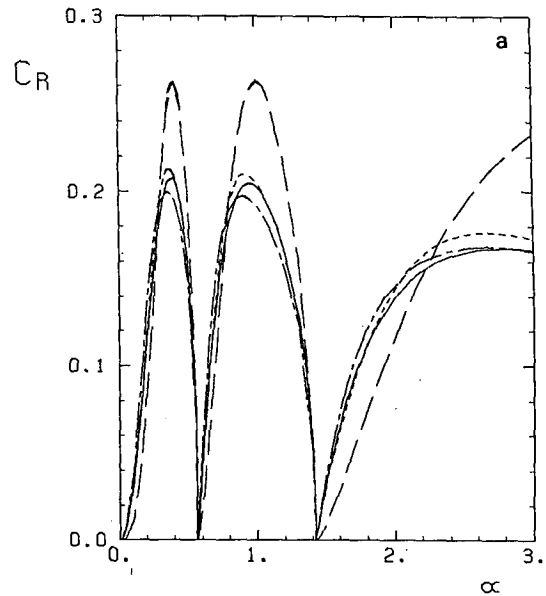


FIG. 6a. As in Fig. 4a except for  $r = 2$ .

largest value of  $\bar{c}_i$  on those terms in (44) which explicitly depend on  $\Delta^2$  is relatively weak. The main dependence of the maximum value of  $\bar{c}_i$  on horizontal wavenumber occurs through the dependence of  $Z_T$  on  $\alpha$  [viz., Eq. (27)]. This is transparently clear in Eq. (45).

Equation (45) reproduces the results initially obtained by Burger (1962) and Miles (1964) for the behavior of the Charney problem in the neighborhood of neutral points. We immediately obtain that

$$\bar{c}_r = \bar{c}_i = 0, \text{ when } \sin(\pi - Z_T) = 0,$$

or, equivalently, when

$$Z_T = \pi(r + 1)/(2\delta) = 0, \pi, 2\pi, \dots, n\pi, \dots, \quad (46)$$

or, when

$$2\delta/(r + 1) = \infty, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots$$

However, from  $\delta = (\alpha^2 + 1/4)^{1/2} \geq 1/2$  we see that  $2\delta/(r + 1) \geq 1/(r + 1)$ . Thus, the number of neutral points increases as  $r$  increases, but for  $r = 1$ , for example, we only have  $\delta = \infty, 1, 1/2$ .

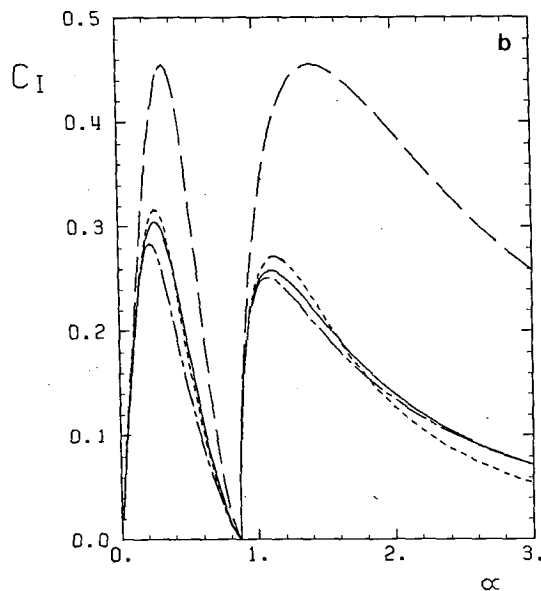


FIG. 5b. As in Fig. 4b except for  $r = 1$ .

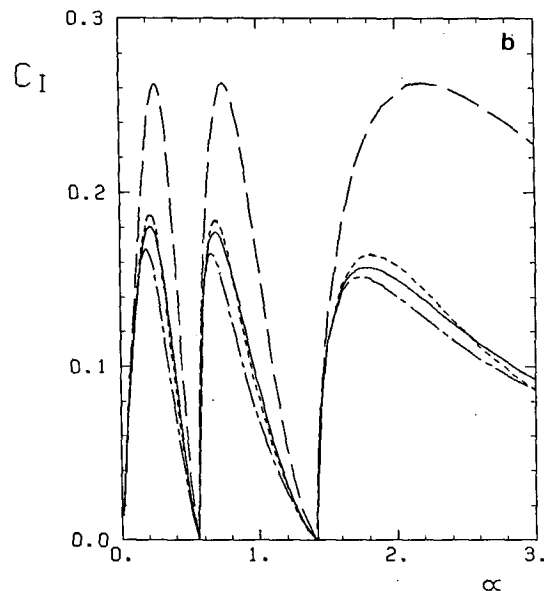


FIG. 6b. As in Fig. 4b except for  $r = 2$ .



Let  $\delta_n \equiv (r + 1)/(2n)$ . In the neighborhood of  $\delta_n$ , let  $\delta = \delta_n + \epsilon$ . From (46),

$$Z_T = \frac{\pi(r + 1)}{2(\delta_n + \epsilon)} \approx \frac{\pi(r + 1)}{2\delta_n} \left(1 - \frac{\epsilon}{\delta_n}\right) \approx n\pi \left(1 - \frac{2n\epsilon}{r + 1}\right)$$

and

$$\pi - Z_T \approx (1 - n)\pi + \frac{2n^2\pi\epsilon}{r + 1}$$

$$\sin(\pi - Z_T) \approx (-1)^{n+1} \frac{2n^2\pi\epsilon}{r + 1}$$

and

$$\cos(\pi - Z_T) \approx (-1)^{n+1}.$$

From (45) we have

$$\tilde{c}^2 \approx \frac{-2/\pi}{r(r + 1)} (-1)^{n+1} \frac{2n^2\pi\epsilon}{r + 1}$$

$$\times \left[ (-1)^{n+1} - i(-1)^{n+1} \frac{2n^2\pi\epsilon}{r + 1} \right]$$

$$\approx \frac{-4n^2\epsilon}{r(r + 1)^2} \left(1 - i \frac{2n^2\pi\epsilon}{r + 1}\right).$$

For  $\epsilon > 0$

$$\tilde{c} \approx \frac{2n}{\sqrt{r(r + 1)}} \epsilon^{1/2} i \left(1 - i \frac{n^2\pi\epsilon}{r + 1}\right)$$

or

$$\tilde{c}_r \approx \frac{2n^3\pi}{\sqrt{r}(r + 1)^2} \epsilon^{3/2}, \quad \tilde{c}_i \approx \frac{2n}{\sqrt{r}(r + 1)} \epsilon^{1/2}. \quad (47)$$

For  $\epsilon < 0$

$$\tilde{c} \approx \frac{2n}{\sqrt{r}(r + 1)} |\epsilon|^{1/2} \left(1 + i \frac{n^2\pi|\epsilon|}{r + 1}\right)$$

or

$$\tilde{c}_r \approx \frac{2n|\epsilon|^{1/2}}{\sqrt{r}(r + 1)}, \quad \tilde{c}_i \approx \frac{2n^3\pi}{\sqrt{r}(r + 1)^2} |\epsilon|^{3/2}. \quad (48)$$

Apart from notation, Eqs. (47) and (48) are exactly the results obtained by Miles (1964). Reference to Figs. 4–6, however, shows that the neighborhood of  $\delta_n$  where (47) and (48) could be accurate must be extremely small. Indeed, the slow growth of  $\tilde{c}_r$  predicted by (47) for  $\epsilon > 0$ , is not really found. It can be shown that the contribution of term 2*b* in Eq. (44) to the determination of  $\tilde{c}_r$  is important for  $\epsilon > 0$ , no matter how small  $\epsilon$  is, and leads to  $\tilde{c}_r$  being proportional to  $\epsilon$  rather than  $\epsilon^{3/2}$ . In other words, Miles' (1964) result for  $\epsilon > 0$  is incorrect.

**6. Conclusion**

By means of a straightforward extension of conventional WKB analysis, we have obtained simple,

illuminating and accurate approximate solutions to the Charney problem for baroclinic instability. Moreover, the solutions are readily extendible to more general profiles. One merely has to replace (27) with the integral appropriate to other distributions of  $\bar{u}$  and/or boundaries. A simple example of such an extension is presented in the Appendix to the present paper.

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APPENDIX

**Baroclinic Instability in the Presence of an Upper Lid**

In Lindzen *et al.* (1980), numerical results were presented for the baroclinic instability of the Charney problem for  $r = 1$ , and for a rigid lid at  $\bar{z} = 4$ . The results obtained for  $\tilde{c}_r$  vs  $\alpha$  and  $\tilde{c}_i$  vs  $\alpha$  are shown in Figs. 7a and 7b.<sup>1</sup> So too are the numeri-

<sup>1</sup> Note that the results shown in Figs. 7a and 7b differ somewhat from those in Fig. 2 of Lindzen *et al.* (1980) for  $0.87 < \alpha < 1.4$ . There were, in fact, convergence difficulties in this region. The present results are more accurate.

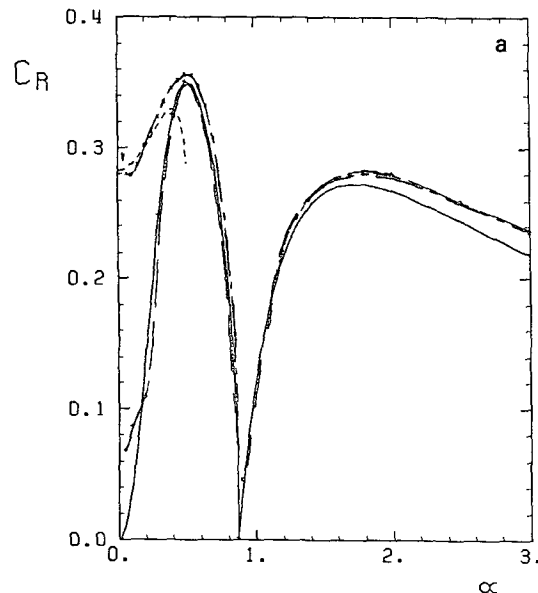


FIG. 7a.  $\tilde{c}_r$  vs  $\alpha$  as obtained numerically when a lid is present at  $\bar{z} = 16$  (—) and at  $\bar{z} = 4$  (---). Also shown are WKB results for a lid (— · —) when  $\alpha \leq 0.5$  (for larger  $\alpha$ 's the turning point occurs below  $\bar{z} = 4$ ) and the WKB results in the absence of a lid (—),  $r = 1$ .

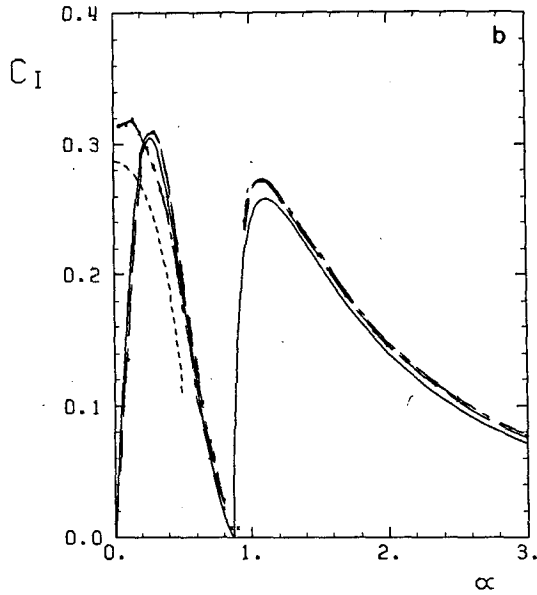


FIG. 7b. As in Fig. 7a except for  $\bar{c}_i$  vs  $\alpha$ .

cal results shown previously in Fig. 1 (obtained with a lid at  $\bar{z} = 16$ ). The results differ pronouncedly for  $\alpha \leq 0.5$ . It is readily shown that, for  $\alpha < 0.5$ , the

turning point described in Section 3 would occur (for  $r = 1$ ) above  $\bar{z} = 4$ . Thus, for  $\alpha \leq 0.5$ , (28) is no longer appropriate for the WKB analysis.

In this Appendix we show how our WKB results can be extended to the case where a rigid lid is present *below* the turning point. The appropriate upper boundary condition in this case is

$$\psi_{\bar{z}} + \frac{1}{2} \psi - \frac{1}{\bar{z} - \bar{c}} \psi = 0, \quad \text{at } \bar{z} = \bar{z}_t = 4, \quad (\text{A1})$$

or, in terms of  $\zeta$ ,

$$\psi_{\zeta} + \frac{1}{2} \psi - \frac{1}{\zeta} \psi = 0, \quad \text{at } \zeta = \zeta_t = \bar{z}_t - \bar{c}. \quad (\text{A2})$$

As in Section 3,

$$Z = \int_0^{\zeta} m d\zeta = \int_0^{\zeta} \left( \frac{r+1}{\zeta'} - \alpha^2 - \frac{1}{4} \right)^{1/2} d\zeta'. \quad (\text{A3})$$

We now define

$$Z_t = \int_0^{\zeta_t - \bar{c}} m d\zeta. \quad (\text{A4})$$

Eq. (A4) is exactly integrable, provided that our lid is below the turning point for  $m^2$ :

$$Z_t = \left( (\bar{z}_t - \bar{c}) \left( \frac{r+1}{\bar{z}_t - \bar{c}} - \alpha^2 - \frac{1}{4} \right)^{1/2} + \frac{1}{2} \frac{(r+1)}{(\alpha^2 + 1/4)^{1/2}} \left[ \frac{\pi}{2} - \sin^{-1} \left[ 1 - 2 \left( \frac{\alpha^2 + 1/4}{r+1} \right) (\bar{z}_t - \bar{c}) \right] \right] \right). \quad (\text{A5})$$

The upper boundary condition is now applied to our WKB solution at  $Z = Z_t$ . From (25) we have

$$\psi \approx \left( \frac{r+1}{\zeta} - \alpha^2 - \frac{1}{4} \right)^{-1/4} \left[ A \exp \left( iZ - \frac{3\pi i}{4} \right) + B \exp \left( -iZ + \frac{3\pi i}{4} \right) \right], \quad (\text{A6})$$

$$\begin{aligned} \psi_{\zeta} \approx \left( \frac{r+1}{\zeta} - \alpha^2 - \frac{1}{4} \right)^{1/4} & \left\{ \left[ i + \frac{r+1}{4\zeta^2} \left( \frac{r+1}{\zeta} - \alpha^2 - \frac{1}{4} \right)^{-3/2} \right] A \exp \left( iZ - \frac{3\pi i}{4} \right) \right. \\ & \left. + \left[ -i + \frac{r+1}{4\zeta^2} \left( \frac{r+1}{\zeta} - \alpha^2 - \frac{1}{4} \right)^{-3/2} \right] B \exp \left( -iZ + \frac{3\pi i}{4} \right) \right\}. \quad (\text{A7}) \end{aligned}$$

Inserting (A6) and (A7) into (A2), we obtain

$$B = - \frac{\left\{ \left[ i + \frac{r+1}{4(\bar{z}_t - \bar{c})^2} f_1^{-3} \right] f_1 + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right\}}{\left\{ \left[ -i + \frac{r+1}{4(\bar{z}_t - \bar{c})^2} f_1^{-3} \right] f_1 + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right\}} \left\{ \exp \left[ 2i \left( Z_t - \frac{3\pi}{4} \right) \right] \right\} A, \quad (\text{A8})$$

where

$$f_1 = \left( \frac{r+1}{\bar{z}_t - \bar{c}} - \alpha^2 - \frac{1}{4} \right)^{1/2},$$

and (A6) becomes

$$\begin{aligned} \psi \approx \left( \frac{r+1}{\zeta} - \alpha^2 - \frac{1}{4} \right)^{-1/4} & \left\{ \left[ \left( -i + \frac{r+1}{4(\bar{z}_t - \bar{c})^2} f_1^{-3} \right) f_1 + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right] \right. \\ & \times \left[ \exp \left( -iZ_t + \frac{3\pi i}{4} \right) \right] \left[ \exp \left( iZ - \frac{3\pi i}{4} \right) \right] \\ & \left. - \left[ \left( i + \frac{r+1}{4(\bar{z}_t - \bar{c})^2} f_1^{-3} \right) f_1 + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right] \left[ \exp \left( iZ_t - \frac{3\pi i}{4} \right) \right] \left[ \exp \left( -iZ + \frac{3\pi i}{4} \right) \right] \right\}. \quad (\text{A9}) \end{aligned}$$

In terms of Hankel functions we obtain, analogously to Eq. (30),

$$\psi \approx C \left( \frac{r+1}{\zeta} - \delta^2 \right)^{-1/4} \sqrt{Z} \times \left( \left[ \left[ -i + \frac{r+1}{4(\bar{z}_t - \bar{c})^2} f_1^{-3} \right] f_1 + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right] \exp \left[ i \left( \frac{3\pi}{4} - Z_t \right) \right] H_1^{(1)}(Z) \right. \\ \left. - \left[ \left[ i + \frac{r+1}{4(\bar{z}_t - \bar{c})^2} f_1^{-3} \right] f_1 + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right] \exp \left[ -i \left( \frac{3\pi}{4} - Z_t \right) \right] H_1^{(2)}(Z) \right). \quad (A10)$$

(A10) is continued to  $\zeta < 0$  exactly as in Section 3, and the dispersion relation is again obtained from the lower boundary condition:

$$\left\{ \frac{1}{2} + \frac{1}{\bar{c}} - \left[ \frac{1}{2X(\bar{c})} \left( \delta^2 + \frac{r+1}{\bar{c}} \right)^{1/2} + \frac{1}{4} \frac{r+1}{\bar{c}^2} \left( \delta^2 + \frac{r+1}{\bar{c}} \right)^{-1} \right] \right\} \\ \times \{ (f_3/f_2) K_1[X(\bar{c})] + I_1[X(\bar{c})] \} - \left( \delta^2 - \frac{r+1}{\bar{c}} \right)^{1/2} \left\{ \frac{f_3}{f_2} K_1'[X(\bar{c})] + I_1'[X(\bar{c})] \right\} = 0 \quad (A11)$$

where

$$f_3(\bar{c}) = -2f_1(\bar{c}) \sin \left( \frac{3\pi}{4} - Z_t \right) + 2 \left[ -\frac{r+1}{4(\bar{z}_t - \bar{c})^2 [f_1(\bar{c})]^2} - \frac{1}{2} + \frac{1}{\bar{z}_t - \bar{c}} \right] \cos \left( \frac{3\pi}{4} - Z_t \right)$$

and

$$f_2(\bar{c}) = \pi i \left\{ -if_1(\bar{c}) + \frac{r+1}{4(\bar{z}_t - \bar{c})^2 [f_1(\bar{c})]^2} + \frac{1}{2} - \frac{1}{\bar{z}_t - \bar{c}} \right\} \exp \left[ i \left( \frac{3\pi}{4} - Z_t \right) \right].$$

Eq. (A11) differs from (35) only insofar as the quantity

$$-[2 \sin(\pi - Z_t)] \pi^{-1} \exp[-i(\pi - Z_t)]$$

is replaced by  $f_3/f_2$ ; the latter, moreover, depends on  $\bar{c}$  in contrast to the former which does not.

Eq. (A11) is solved exactly as (35) was in Section 4. The results are shown in Figs. 7a and 7b. The WKB results form a good approximation right up to the point where the turning point and lid coincide. Beyond this point the solutions for an unbounded atmosphere are quite adequate.

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