3 Large-scale flow on the rotating Earth
I: Shallow water dynamics

3.1 Shallow water equations with rotation

The basic, inviscid, equations are

\[
\frac{du}{dt} - fv = -\frac{\partial h_s}{\partial x}
\]
\[
\frac{dv}{dt} + fu = -\frac{\partial h_s}{\partial y}
\]
\[
\frac{dh}{dt} + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0
\]

where \( f = 2\Omega \sin \varphi \), with \( \varphi \) being latitude. Here, \( h \) is the depth of the fluid; \( h_s \) is the height of the free surface; so, if the base of the fluid layer is at height \( h_b \), \( h = h_s - h_b \). (For reasons that will become apparent later, we want to allow the possibility that the base is not flat.)

Initially, however, let’s focus on a system with a flat base, \( h_b = 0 \), so that \( h = h_s \). Assume a motionless basic state with uniform depth \( h = D \), with small-amplitude perturbations such that \( u = u', v = v', h = D + h' \), and linearize to get

\[
\frac{\partial u'}{\partial t} - f v' = -\frac{\partial h'}{\partial x}
\]
\[
\frac{\partial v'}{\partial t} + f u' = -\frac{\partial h'}{\partial y}
\]
\[
\frac{\partial h'}{\partial t} + D \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = 0
\]

3.1.1 \( f \)-plane solutions

Suppose \( f \) is constant. Then we look for solutions

\[
\begin{pmatrix}
u' \\
v' \\
h'
\end{pmatrix} = \text{Re} \begin{pmatrix} U \\ V \\ H \end{pmatrix} e^{i(kx+ly-\omega t)}
\]
whence

\[-i\omega U - fV + ikgH = 0\]
\[-i\omega V + fU + iglH = 0\]
\[-i\omega H + D (i kU + i lV) = 0\]

which leads to the dispersion relation

\[\omega \left[\omega^2 - f^2 - (k^2 + l^2)gD\right] = 0.\] (3)

The two nonzero roots of (3), \(\omega = \pm \sqrt{f^2 + (k^2 + l^2)gD}\) are interia-gravity waves, which have the familiar limits \(\omega = \pm \sqrt{k^2 + l^2}gD\) (surface gravity waves) for \(f \to 0\), and \(\omega = \pm f\) (inertial waves) for \(D \to 0\). If we associate \(D\) with the height of the tropopause, then the periods implied by (3) are much too short (<1day) to be of much interest on the large scale. So the only relevant solution to us here is \(\omega = 0\), which simply implies nondivergent flow in geostrophic balance:

\[-fv' = -g\frac{\partial h'}{\partial x}\]
\[+fu' = -g\frac{\partial h'}{\partial y}\]
\[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0\]

### 3.2 \(\beta\)-plane dynamics

In reality \(f = 2\Omega \sin \varphi\) is not constant, but varies in the \(y\) direction. So we allow \(f = f(y)\); for reasons to become apparent later, we will also allow for the possibility of a base that is not flat, so we allow \(h_b = h_b(y)\) also.

#### 3.2.1 Potential vorticity conservation

We want to form the curl of the momentum equations, the first two of (1). First we note that

\[\frac{\partial}{\partial y} \left( \frac{du}{dt} \right) = \frac{d}{dt} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),\]
\[\frac{\partial}{\partial x} \left( \frac{dv}{dt} \right) = \frac{d}{dt} \left( \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),\]
so
\[ \frac{d\zeta}{dt} + (f + \zeta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + v \frac{df}{dy} = 0, \]

where
\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]
is the vertical component of the vorticity. Since \( f = f(y) \), \( df/dt = v \frac{df}{dy} \), so we have
\[ \frac{d\zeta_a}{dt} + \zeta_a \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \]

where \( \zeta_a = f + \zeta \) is the absolute vorticity (the vertical component of vorticity relative to an inertial frame; thus we identify the Coriolis parameter as the vertical component of the planetary vorticity).

Now, from the third of (1),
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{h} \frac{dh}{dt}, \]

so we finally get
\[ \frac{d\Pi}{dt} = 0 \]

(4)

where
\[ \Pi = \frac{\zeta_a}{h} \]

is the potential vorticity (PV) for our shallow water system. (4) states that, for our inviscid shallow water flow, PV is conserved.

If we had done the analysis including a flat rigid lid and flat base, \( h \) would be constant and then (4) is a statement that absolute vorticity is conserved by fluid columns as they move around. So if, for example, a fluid column in a fluid of fixed depth is moved northward (so \( f \) increases), \( \zeta \) must decrease to conserve \( \zeta_a \) — the fluid element acquires anticyclonic vorticity. In the more general case of varying \( h \), it is \( \Pi \), and not \( \zeta_a \) that is conserved: in addition to the effects of varying \( f \), the absolute vorticity of fluid columns increases (or decreases) as they stretch (or compress).

### 3.2.2 The quasigeostrophic case

For small Rossby number \( Ro = U/fL \), where \( U \) and \( L \) are typical velocity and length scales, and for small \( U/T \), where \( T \) is the time scale (if we assume
\( T \sim L/U \), this is the same thing as small \( Ro \), the equations of motion reduce to geostrophic balance:

\[
\begin{align*}
fu &= -g \frac{\partial h_s}{\partial y} \\
fv &= g \frac{\partial h_s}{\partial x}
\end{align*}
\]  

(6)

from which we deduce that

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{g}{f^2} \frac{\partial h_s}{\partial x} \frac{df}{dy} = \frac{1}{f} \frac{df}{dy}.
\]

Now, the scale for each term on the left is \( U/L \); the scale for the term on the right is \((U/L) (\Delta f/f)\), where \( \Delta f \) is the change in \( f \) over the length scale \( L \). Provided \( \Delta f/f \ll 1 \), therefore, the geostrophic flow is, to leading order, nondivergent:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\]

(7)

Note that

\[
\frac{L}{f} \frac{df}{dy} = \frac{L}{2\Omega a \sin \phi} \frac{d}{d\phi} (2\Omega \sin \phi) = \frac{L}{a} \cot \phi,
\]

so that we require \( L \ll a \): the length scale\(^1\) is much less than the planetary radius. This being the case, we can expand, near latitude \( \phi_0 \),

\[
f = 2\Omega \sin \phi \approx 2\Omega \sin \phi_0 + 2\Omega \cos \phi_0 (\phi - \phi_0)
\]

\[
\approx f_0 + \beta y
\]

where

\[
\beta = \frac{2\Omega}{a} \cos \phi_0
\]

and \( y = a(\phi - \phi_0) \). Note that \( |\beta y| \ll f_0 \), by our assumption, so that (6) allows us to write

\[
u = -\frac{\partial \psi}{\partial y}; \quad v = \frac{\partial \psi}{\partial x},
\]

(8)

where

\[
\psi = \frac{g \delta h_s}{f_0}
\]

(9)

\(^1\)Actually it is the length scale in latitude that matters here.
is the \textit{geostrophic streamfunction}, which guarantees that the geostrophic flow is nondivergent. Here $\delta h_s = h_s - H_0$, where $H_0$ can be any constant (since (8) depends only on derivatives of $\psi$), usually taken to be the average height of the surface. Since $u$ is normal to $\nabla \psi$, it is directed along contours of $\psi$, as shown in Fig. 1.

![Contour of $\psi$](image)

Figure 1: Flow is along streamlines (lines of constant $\psi$ and $\delta h$).

Note that, in terms of streamfunction, the vorticity can be written

$$
\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \equiv \nabla^2 \psi
$$

where $\nabla^2$ is the two-dimensional (horizontal) Laplacian operator.

For perturbations to a motionless background state, such that, for small $Ro$, $|\zeta| \ll f_0$, and, if $h = D_0 + \delta h$ where $D_0$ is constant and $|\delta h| \ll D_0$, the PV is

$$
\Pi = \frac{f + \zeta}{D_0 + \delta h} \approx \frac{f_0 + \beta y}{D_0} + \frac{\zeta}{D_0} - f_0 \frac{\delta h}{D_0} = \Pi_p + \frac{1}{D_0} \left( \zeta - f_0 \frac{\delta h}{D_0} \right)
$$

where $\Pi_p = f(y)/D_0$ is the (unchanging) planetary PV based on the constant reference depth. Since $\delta h = \delta h_s - h_b$, we can write this as

$$
\Pi \approx \Pi_p + \frac{f_0 h_b}{D_0^2} + \frac{1}{D_0} \left( \zeta - f_0 \frac{\delta h_s}{D_0} \right).
$$

Hence, in terms of streamfunction, using (6) and (10), the contribution to PV due to the flow is

$$
\zeta - f_0 \frac{\delta h_s}{D_0} = \nabla^2 \psi - \frac{1}{L_R^2} \psi
$$
where

$$L_R = \frac{\sqrt{gD}}{f_0}$$

(11)
is the Rossby radius of deformation. Thus, in the QG case, the PV is

$$\Pi = \frac{1}{D_0} \left[ f_0 \left( 1 + \frac{h_b}{D_0} \right) + \beta y + \nabla^2 \psi - \frac{1}{L_R^2} \psi \right].$$

(11)

Usually, the constant multiplier $D_0^{-1}$ is removed and the quasigeostrophic potential vorticity for the shallow water case is defined as

$$q = D_0 \Pi = f_0 \left( 1 + \frac{h_b}{D_0} \right) + \beta y + \nabla^2 \psi - \frac{1}{L_R^2} \psi$$

(12)

As we have seen, under conservative conditions PV is simply advected by the flow, so that knowing the flow allows one in principle to predict PV. One of the advantages of QG theory is that, as (12) makes clear, one can in turn close the problem and determine the flow, given the PV, by inverting (12) to determine $\psi$.

### 3.2.3 PV inversion: flow around a point vortex

Suppose there is a point (potential) vortex in a system with a flat base $(h_b = 0)$, for which $q(x, y) = f_0 + \beta y + Z_0 \delta(x - x_0) \delta(y - y_0)$ [so $q = f$ everywhere except at $(x_0, y_0)$]. Since we can anticipate the problem to have circular symmetry, we move into polar coordinates $(r, \theta)$, with $(x_0, y_0)$ as the origin (see Fig. 2). In polar coordinates, the Laplacian is

$$\nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

so that, if we look for symmetric solutions for which $\psi = \psi(r)$ then, everywhere except $r = 0$, $\nabla^2 \psi - L_R^{-2} \psi = 0$ or

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) - \frac{1}{L_R^2} \psi = 0.$$

The solution bounded as $r \to \infty$ is

$$\psi = AK_0 \left( \frac{r}{L_R} \right)$$
where $A$ is a constant, and $K_0(x)$ is the zero-order modified Bessel function. This function, which decays monotonically away from the origin, behaves as $-\ln x$ for $x \ll 1$ and as $(\pi/2x)^{1/2} \exp(-x)$ for $x \gg 1$. Note that the azimuthal velocity, for small $r/L_R$, is

$$u = -\frac{d\psi}{dr} = \frac{A}{r}.$$  

To determine $A$, we note that the circulation\(^2\) around any contour $C$ closely enclosing the point vortex is\(^3\)

$$C = \oint_C \zeta \, dA = \iint_Z \delta(x-x_0)\delta(y-y_0) \, dx \, dy = Z_0.$$  

\(^2\)The circulation around a closed horizontal contour $C$ is defined to be

$$C = \oint_C \mathbf{u} \cdot d\mathbf{l}$$  

where the integral around the contour. Since $\zeta = \hat{z} \cdot \nabla \times \mathbf{u}$, where $\hat{z}$ is the vertical unit vector, it follows directly from Stokes' theorem that

$$C = \iint_C \zeta \, dA$$  

where the integral is over the area enclosed by the contour $C$.  

\(^3\)Note that the fluid depth must be finite (and therefore $\psi$ must be finite) at the origin. Therefore the singularity at $r = 0$ must be entirely in the vorticity.
But if we choose a circular contour at radius $r \ll L_R$, then

$$C = \oint_C \mathbf{u} \cdot d\mathbf{l} = \int_0^{2\pi} u(r) \ r \ d\theta = 2\pi r \ u = 2\pi A,$$

and so $A = Z_0/(2\pi)$. Hence the solution is

$$\psi(r) = \frac{Z_0}{2\pi} K_0 \left( \frac{r}{L_R} \right), \hspace{1cm} (13)$$

as illustrated schematically in Fig. 3.

![Figure 3: Circulation around a cyclonic point vortex (northern hemisphere).](image)

One important property of fluid flow—and rotating flow in particular—that this example makes clear is that the circulation is nonlocal: even a localized vorticity will induce a remote circulation, just as electrical charges induce a remote field. Amongst other things, this means that one cannot in general think about fluid dynamics in terms of local, fluid parcel arguments, since the flow at the location of the parcel depends on the behavior of all other parcels. To be more specific, note from (13) that the length scale characterizing the reach of the induced circulation is just $L_R$: this is the significance of the deformation radius.

### 3.2.4 The barotropic case

If, instead of a free surface, our shallow water system has a flat rigid lid and flat base, such that the fluid has a fixed depth $D$, then the flow is nondivergent
(since \(dh/dt = 0\)) and the PV, in the non-QG case, is just \(\hat{\zeta}_a/D\). Hence conservation of PV implies conservation of absolute vorticity. Under QG assumptions, the barotropic limit of (12) is therefore obtained in the limit \(L_R \to \infty\). So, around a point vortex, \(\psi \sim \ln r\); barotropic flow has no preferred length scale.

3.3 Rossby waves on a shallow-water \(\beta\)-plane

3.3.1 Waves on a zonal flow

If we linearize about a basic state with uniform, purely zonal, flow \((u_0, v_0) = (u_0,0)\), the linearized perturbation PV equation, from (4) and using QG assumptions so that \(\Pi = D_0^{-1}q\),

\[
\frac{dq}{dt} = 0 \quad \Rightarrow \quad \frac{\partial q'}{\partial t} + u_0 \frac{\partial q'}{\partial x} + v' \frac{dq_0}{dy} = 0 ,
\]

where, as usual, primed quantities are perturbations from the basic state (assumed small), and \(q_0\) is the QGPV of the basic state. In terms of the streamfunction

\[
\left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \left( \nabla^2 \psi' - \frac{\psi'}{L_R^2} \right) + \frac{\partial \psi'}{\partial x} \frac{dq_0}{dy} = 0 ,
\]

(14)

Note that the basic state PV just appears in (14) in the form of its gradient which, using (12) with \(\psi_0 = -u_0 y\), is

\[
\frac{dq_0}{dy} = \beta + \frac{f_0}{D} \frac{\partial h_b}{\partial y} + \frac{u_0}{L_R^2} \equiv \hat{\beta} ,
\]

(15)

(15)

There is a curiosity here: addition of a uniform \(u_0\) does not just introduce a Galilean transformation \(\omega \to \omega - ku_0\), but there is also a “non-Doppler” term hidden in the definition of \(\hat{\beta}\). This is a consequence of the representation of the sphere by the \(\beta\)-plane in a shallow water model. In practice, \(\hat{\beta} \approx \beta\), so we will not dwell on the issue here.
The “intrinsic frequency” $\tilde{\omega}$ is plotted in Fig. 4. The phase velocity, $c = \tilde{\omega}/k$, is always negative, relative to the mean flow (since $\beta > 0$ everywhere on the sphere): Rossby wave phase always propagates westward (relative to the background flow). Also plotted schematically is the intrinsic group velocity, which has components

$$\frac{\partial \tilde{\omega}}{\partial k} = \frac{\partial \omega}{\partial k} - u_0 = \frac{\hat{\beta} (k^2 - l^2 - L_R^{-2})}{(k^2 + l^2 + L_R^{-2})^2},$$

$$\frac{\partial \tilde{\omega}}{\partial l} = \frac{\partial \omega}{\partial l} = \frac{2\hat{\beta}kl}{(k^2 + l^2 + L_R^{-2})^2}.$$

Figure 4: Plot showing contours of $\tilde{\omega}/\hat{\beta}L_R$ as a function of $kL_R$ and $lL_R$. Arrows indicate direction of group velocity.
Note that these can have either direction. In the zonal direction, group velocity relative to the background flow can (unlike phase velocity) have either sign: the long waves \((k^2 < l^2 + L_R^{-2})\) propagate westward, the short waves eastward (relative to the background flow).

### 3.3.2 Typical numbers

For the atmosphere, we assume \(h_b = 0\). At 45 degN, \(f = 1.03 \times 10^{-4} \text{s}^{-1}\), so if \(D = 10\text{km}\)

\[
L_R = \frac{\sqrt{gD}}{f} = 3.0 \times 10^6 \text{ m},
\]

while

\[
\beta = \frac{2\Omega}{a} \cos \frac{\pi}{4} = 1.6 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1},
\]

so, for a mean wind of 20ms\(^{-1}\),

\[
\hat{\beta} = \beta + \frac{u_0}{L_R^2} = 1.6 \times 10^{-11} + \frac{20}{9 \times 10^{12}} \approx 1.8 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}
\]

(note that \(\hat{\beta}\) is not too different from \(\beta\)). A typical midlatitude disturbance might have a half-wavelength of 4000km in both directions, so \(k = l \approx \pi / (4 \times 10^6) = 7.85 \times 10^{-7} \text{m}^{-1}\), and then the westward phase speed, relative to the background flow, is

\[
c - u_0 = \frac{\omega}{k} = -\frac{\hat{\beta}}{k^2 + l^2 + L_R^2} = \frac{1.8 \times 10^{-11}}{1.3436 \times 10^{-12}} = -13.4 \text{ ms}^{-1};
\]

the intrinsic frequency is

\[
\tilde{\omega} = -k(c - u_0) = 1.05 \times 10^{-5} \text{s}^{-1},
\]

and so the intrinsic period is

\[
\frac{2\pi}{|\tilde{\omega}|} = \frac{2\pi}{1.05 \times 10^{-5}} = 6.0 \times 10^5 \text{s} = 6.9 \text{ d}.
\]

So the typical periods and phase speeds (relative to the background flow) for these \textit{planetary scale} Rossby waves are of order (days) and comparable with wind velocities, and so are meteorologically significant.
3.3.3 Mechanism of Rossby wave propagation

From (15), it is clear that the propagation of Rossby waves (indeed, the existence of the waves themselves) is dependent on the existence of the planetary vorticity gradient, $\beta$. In fact, had we allowed the basic state to have relative vorticity, it would have been the gradient of the mean absolute vorticity, rather than just $\beta$, that appeared in (15). How does a basic state PV gradient lead to waves? Consider Fig. 5. We assume that there are two regions of uniform PV, separated initially by a straight E-W boundary. North of the boundary, the QGPV is $q_2$; to the south, it is $q_1$. Since the Coriolis parameter increases northward, we specify that $q_1 < q_2$. Now let’s perturb the interface by applying a local northward displacement as shown on the figure. Because of potential vorticity conservation, there is now a perturbation in the PV field, which is zero everywhere except in the bulge in the interface, where the PV perturbation is $q_1 - q_2 < 0$: the anomaly is negative. Just as perturbing an array of electric charges would induce an anomalous electric field, this PV perturbation will induce an anomalous circulation. In fact, the streamfunction of the perturbed circulation is $\psi'$ where $\nabla^2 \psi' - \psi / L_R^2 = q'$. So the problem of determining the circulation is essentially the same as that for the circulation around a potential point vortex: the induced circulation will

![Figure 5: Schematic of induced flow around a perturbed PV interface. Note that $\Pi = D_0^{-1}q$.](image)
be clockwise, decaying as $K_0 (r/L_R)$ from the PV anomaly, much as depicted schematically in the figure.

Now, because PV is conserved following the flow, it is simply advected by the circulation. The effect of the induced circulation on the PV distribution will be to advect the interface as shown by the dashed lines: northward to the west, southward to the east. As the initial perturbation was northward, the perturbation itself tends to move toward the west—this is the westward phase propagation we noted from (15). The spreading, and changing of shape of the perturbation—evident, amongst other things, by the developing southward perturbation to the east—is a manifestation of the dispersion we also noted.

### 3.4 Rossby waves in a fluid of varying depth

Consider now perturbations to a fluid contained between sloping surfaces, as in Fig. 3.4. The column depth, $H(y)$ is a linear function of $y$ (but its variations are assumed small), and we are now considering an $f$-plane (constant $f$). Then the basic state PV gradient is

$$H \frac{d\Pi_0}{dy} = H \frac{d}{dy} \left( \frac{f}{H} \right) \sim -\frac{f}{H} \frac{dH}{dy},$$

so (15) is still valid if we simply identify $\hat{\beta}$ as

$$\hat{\beta} = -\frac{f}{H} \frac{dH}{dy}. \quad (16)$$

Thus, the vorticity equation becomes precisely equivalent to that in the free surface case on a $\beta$-plane, with in this case $\hat{\beta}$—a measure of the gradient of fluid depth—replacing the gradient of $f$. Thus, e.g., a sloping bottom in
the ocean — or in the laboratory\(^5\) — can give rise to Rossby waves, called “topographic Rossby waves,” just as can the curvature of the Earth.

In fact, in the case of the Earth’s curvature, the two effects are just another way of saying the same thing. Each is illustrated in Fig. 6. On

![Figure 6: Illustrating the equivalence between the two forms of “beta” in spherical geometry.](image)

On the left, we take a traditional view of the atmosphere (or ocean), which is assumed to be contained within a spherical shell of depth \(D\). The “vertical” is defined to be the local upward normal to the surface, and the component of planetary vorticity in this direction is \(2\Omega \sin \phi = f\), the Coriolis parameter. Since the thickness of the fluid in the vertical direction is \(D\), the potential vorticity is

\[
P = \frac{f}{D} = \frac{2\Omega \sin \phi}{D},
\]

and its gradient is

\[
\frac{1}{a} \frac{dP}{d\phi} = \frac{2\Omega}{aD} \cos \phi = \frac{1}{D} \frac{df}{dy} = \frac{\beta}{D}.
\]

In this view, the depth of the fluid column, \(D\), never changes, so conservation of potential vorticity \(\Pi\) implies conservation of absolute vorticity \(\zeta_a\). If a fluid

\(^5\)This is, of course, achievable, whereas a spatially varying \(f\) is not.
column is moved northward to where $f$ is greater, $\zeta_a = f + \zeta$ is conserved by $\zeta$ decreasing as $f$ increases—so a northward displacement induces anticyclonic (negative) relative vorticity.

In the second view, we define the direction of the Earth’s rotation vector to be the “vertical.” The component of planetary vorticity in this direction is just $2\Omega$, which is of course constant. But the thickness of the atmospheric shell in this is not constant, but is $h_0 = D/\sin \phi$. So the potential vorticity is

$$\Pi = \frac{\zeta_a}{h_0} = \frac{2\Omega \sin \phi}{D},$$

just the same! And its gradient is

$$\frac{1}{a} \frac{dP}{d\phi} = \frac{2\Omega}{a} \frac{d}{d\phi} \left( \frac{1}{h_0} \sin \phi \right) = \frac{2\Omega}{a} \frac{d}{d\phi} \left( \frac{\sin \phi}{D} \right) = \frac{\beta}{D}.$$

So the PV gradient is (of course) exactly the same as in the first case, but we see it differently. In this viewpoint, the planetary vorticity is everywhere $2\Omega$, but as fluid columns move north or south, their length changes. A northward displacement produces a contraction of the column: in response (in order to conserve $\Pi$) the absolute vorticity $2\Omega + \zeta$ must decrease, so $\zeta$ must become anticyclonic (negative).

### 3.4.1 Lab experiment: topographic Rossby waves in the lee of a ridge

Fig 7 show the set-up. A cylindrical tank, on a turntable rotating at rate $\Omega$, is fitted with a conical base; since the deepest water is at the outer rim, that corresponds to the equator. The effective $\beta$ in this setup is

$$\tilde{\beta} = \frac{2\Omega}{H} \frac{dH}{dr} = \frac{2\Omega}{R} \frac{\delta H}{H}.$$

A lid rotates cyclonically relative to the tank at rate $\omega$. This drives flow (of angular velocity $\sim \omega/2$) in the tank, over a small, straight ridge on the conical base. Any waves produced by the stationary ridge will be stationary, i.e., $\tilde{\omega} = 0$. Hence, from (15)$^6$,

$$-ku_0 = -\frac{k\tilde{\beta}}{k^2 + l^2}.$$

$^6$In this case, the upper surface is rigid, so $\delta h = 0$; this leads to the absence of the term $L_R^2$ in (15). [This is the barotropic case.]
and so we expect this to produce a train of stationary Rossby waves of total wavenumber

\[ \kappa = \sqrt{k^2 + l^2} \approx \frac{\beta}{U} \]

where \( U = (R/2) (\Delta \Omega / 2) \) is the flow at radius \( R/2 \). So we expect the magnitude of the wavelength to be

\[ \frac{2\pi}{\kappa} = 2\pi \sqrt{\frac{U}{\beta}} = \pi \sqrt{\frac{\Delta \Omega R^2}{2\Omega}} \left( \frac{H}{\delta H} \right) = \pi R \times \sqrt{\left( \frac{\Delta \Omega}{2\Omega} \right) / \left( \frac{\delta H}{H} \right)} . \]

In the experiment, \( \Delta \Omega \approx 0.1\Omega \), and \( \delta H \approx H/2 \), so we expect

\[ \frac{2\pi}{\kappa} \approx \pi R \times 0.15 . \]

Since, at mid-channel, a wave of zonal wavenumber one has wavelength \( \pi R \), this will give us something like zonal wavenumber 6, as seen in the movie of the experiment.
II: Dynamics in a rotating, stratified atmosphere

3.5 QG motions in the atmosphere

Our full, log-pressure, equations for inviscid, adiabatic motions on a \( \beta \)-plane in the (compressible) atmosphere are:

\[
\frac{du}{dt} - fv = - \frac{\partial \phi}{\partial x} \\
\frac{dv}{dt} + fu = - \frac{\partial \phi}{\partial y} \\
\frac{d\theta}{dt} = 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) = 0 \\
\frac{\partial \phi}{\partial z} = \frac{\kappa}{H} \bar{\Pi}(z) \theta
\]  

(17)

We define geostrophic velocities

\[
v_g = \frac{1}{f_0} \frac{\partial \phi}{\partial x}; \quad u_g = - \frac{1}{f_0} \frac{\partial \phi}{\partial y}
\]

(18)

(and, implicitly, \( w_g = 0 \)). So the geostrophic streamfunction is just \( \phi / f_0 \). Using the last of (17), the corresponding thermal wind shear equations are

\[
\frac{\partial v_g}{\partial z} = \frac{\kappa}{f_0 H} \bar{\Pi}(z) \frac{\partial \theta}{\partial x}; \quad \frac{\partial u_g}{\partial z} = - \frac{\kappa}{f_0 H} \bar{\Pi}(z) \frac{\partial \theta}{\partial y}.
\]

(19)

We now assume that the ageostrophic velocities \( u_a = u - u_g, v_a = v - v_g, w_a = w \), are small compared with these. Then, in the advection terms (which are small under QG scaling) we may approximate

\[
\frac{du}{dt} \simeq D_g u_g, \quad \frac{dv}{dt} \simeq D_g v_g, \quad \frac{d\theta}{dt} \simeq D_g \theta + w_a \frac{d\theta_0}{dz}
\]

where

\[
D_g \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}
\]

is the advective derivative following the geostrophic flow.

To proceed to a full set of QG equations, we must make a decision about the scaling of the potential temperature gradients. In adiabatic flow, the
motions must be along isentropes (in order to conserve $\theta$). The isentropic slope has magnitude $\varepsilon = |\nabla_h \theta| / (d\theta/dz)$; therefore the vertical and horizontal velocity components must obey $W/U \sim \varepsilon$, where $U, W$ are the scales for horizontal and vertical velocities. Given that the divergence of the ageostrophic horizontal flow must be $O(Ro \times U/L)$, we have $W \sim Ro HU/L$. Therefore the isentropic slope must satisfy

$$\varepsilon \sim \frac{Ro}{L}.$$

The implication of this is that $Hd\theta/dz \sim Ro^{-1} L |\nabla_h \theta|$, so that vertical variations in $\theta$ exceed the horizontal variations by a factor $Ro^{-1}$. Putting it another way, to leading order $\theta$ is a function of $z$ only. (Horizontal variations are weaker, but important, as we’ll see.) So, in the thermodynamic equation, we must resist the temptation of assuming that vertical advection of $\theta$ is small (on the grounds that vertical velocity is small), since

$$w_a \frac{d\theta_0}{dz} / |\mathbf{u}_g \cdot \nabla \theta| \sim \frac{w_a L}{U H Ro} \sim O(1),$$

since $w_a \sim \varepsilon U \sim Ro U H/L$.

Then we have, with $f = f_0 + \beta y$ (and $|\beta y| \ll f_0$)

$$D_g u_g - f_0 v_a - \beta y v_g = 0$$

$$D_g v_g + f_0 u_a + \beta y u_g = 0$$

$$D_g \theta + w_a \frac{d\theta_0}{dz} = 0$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w_a) = 0$$

$$\frac{\partial \phi}{\partial z} - \frac{\kappa}{H} \bar{\Pi}(z) \theta = 0$$

where $\theta_0(z)$ is the horizontally averaged potential temperature. Now, from the curl of the first two of (20), we can form the vorticity equation\(^7\)

$$D_g \xi_g + \beta v_g + f_0 \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) = 0,$$

---

\(^7\)Note that

$$\frac{\partial}{\partial x} (D_g v_g) - \frac{\partial}{\partial y} (D_g u_g) = D_g \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) \left( \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right),$$

and the last set of terms vanishes because the geostrophic flow is nondivergent. Then (21) follows.
where \( \zeta_g = \partial v_g / \partial x - \partial u_g / \partial y \) is the geostrophic vorticity. From the continuity and thermodynamic eqs., and since \( \theta_0 = \theta_0(z) \) and \( \rho = \rho(z) \), we have

\[
\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = -\frac{1}{\rho} \frac{\partial}{\partial z} (\rho w_a) = \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho D_g (\theta - \theta_0) \right) = \frac{1}{\rho} \frac{\partial}{\partial z} \left[ D_g \left( \frac{\theta - \theta_0}{\rho \partial \theta_0 / \partial z} \right) \right].
\]

Now,

\[
\frac{\partial}{\partial z} \left[ D_g \left( \frac{\theta - \theta_0}{\rho \partial \theta_0 / \partial z} \right) \right] = D_g \left[ \frac{\partial}{\partial z} \left( \frac{\theta - \theta_0}{\rho \partial \theta_0 / \partial z} \right) \right] + \frac{\rho}{\partial \theta_0 / \partial z} \left( \frac{\partial u_g \partial \theta}{\partial x} + \frac{\partial v_g \partial \theta}{\partial y} \right).
\]

Using the thermal wind shear relations, the second term is zero. Hence

\[
\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = D_g \left[ \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\theta - \theta_0}{\rho \partial \theta_0 / \partial z} \right) \right]
\]

and so

\[
D_g \zeta_g + \beta v_g + f_0 D_g \left[ \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\theta - \theta_0}{\rho \partial \theta_0 / \partial z} \right) \right] = 0.
\]

Using \( v_g \beta = D_g f \), as before, we then have

\[
D_g q = 0,
\]

where

\[
q = f_0 + \beta y + \zeta_g + \frac{1}{\rho} \frac{\partial}{\partial z} \left( f_0 \rho \frac{\theta - \theta_0}{\rho \partial \theta_0 / \partial z} \right)
\]

is the quasigeostrophic potential vorticity (QGPV; sometimes known as pseudo-PV). (22) states that, for inviscid, adiabatic flow, QGPV is conserved following the geostrophic flow.

### 3.6 Baroclinic Rossby waves on zonal flow

(Some related material: Holton, Section 12.3)

Let’s now look at small-amplitude perturbations to a basic state with uniform zonal flow \( u_0 \), potential temperature \( \theta_0(z) \) (with \( N^2 \) constant) and QGPV \( q_0 = f = f_0 + \beta y \). Then, linearizing about that state, we get the perturbation QGPV eq.

\[
\left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) q' + v' \frac{\partial q_0}{\partial y} = 0.
\]
In terms of geostrophic streamfunction $\psi' = \phi'/f_0$, we have $v' = \partial \psi'/\partial x$, $u' = -\partial \psi'/\partial y$, $\theta' = \left( fH/\kappa \bar{\Pi} \right) \partial \psi'/\partial z$, and

$$ q' = \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial}{\partial z} \right) \right] \psi' $$

where $N^2 = \kappa \bar{\Pi}(z) \left( d\theta_0/dz \right)/H$. Then

$$ \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \left\{ \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial}{\partial z} \right) \right] \psi' \right\} + \beta \frac{\partial \psi'}{\partial x} = 0 \quad (24) $$

All the coefficients in (24) except for $\rho$ are constant (by assumption). As before, we deal with $\rho(z) = \rho_0 \exp(-z/H)$ by anticipating

$$ \psi'(x, y, z, t) = \rho_0 \exp \left( \frac{z}{2H} \right) \exp \left( \frac{z}{2H} \right) $$

whence

$$ \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{\partial \psi'}{\partial z} \right) = \left( \frac{\partial^2 \psi'}{\partial z^2} - \frac{1}{4H^2} \psi' \right) \exp \left( \frac{z}{2H} \right) $$

so that

$$ \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \left\{ \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{f_0^2}{N^2} \left( \frac{\partial^2}{\partial z^2} - \frac{1}{4H^2} \right) \right] \psi' \right\} + \beta \frac{\partial \psi'}{\partial x} = 0 \quad (24) $$

This now has constant coefficients, and has solutions

$$ \Psi' = \text{Re} \left\{ \Psi_0 \exp \left[ i (kx + ly + mz - kct) \right] \right\} $$

where

$$ (c - u_0) \left[ k^2 + l^2 + \frac{f_0^2}{N^2} \left( m^2 + \frac{1}{4H^2} \right) \right] + \beta = 0 \quad (25) $$

This constitutes our dispersion relation for baroclinic, QG, Rossby waves on a uniform flow with constant $N^2$.

**3.6.1 Vertical modes**

Does the atmosphere have modes? It is a bounded system (with finite mass). Clearly, it is bounded in the horizontal (by the finiteness of the sphere), but
what about vertical structure? To address this, we go back to (24), with 
\( u_0 = \text{constant and } N^2 \text{ uniform, and separate the } z\text{-variation} \)

\[
\psi' = Z(z)F(x,y,t)
\]

so that normal separation-of-variables procedures lead us to

\[
\left[ \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) F \right]^{-1} \left[ \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \beta \frac{\partial F}{\partial x} \right) \right] = -f_0^2 \frac{gh_e}{g}
\]

where \( f_0^2/gh_e \) is the separation constant. The first equality has solutions

\[
Z \sim \exp (\mu z)
\]

where \( \mu \) has the constant value defined by

\[
\mu \left( \mu - \frac{1}{H} \right) = -\frac{N^2}{gh_e}
\]

i.e.,

\[
\mu = \frac{1}{2H} \pm \sqrt{\frac{1}{4H^2} - \frac{N^2}{gh_e}}.
\]

Now, we need to apply boundary conditions. As \( z \to \infty \), we require boundedness of the disturbance energy \( \rho u'^2 \), which requires that \( \text{Re}(\mu - \frac{1}{2H}) < 0 \); accordingly, we must have \( N^2/gh_e < 1/4H^2 \) and choose the negative root. At the ground, we need to be careful. The actual surface is flat, but we are using log-pressure coordinates, and surface pressure may vary, so we cannot assume that \( w = -(H/p)dp/dt \) is zero there. However, since for our case \( \partial u_0/\partial z = 0 \), the basic state has no horizontal temperature gradients; in particular (since \( p \) is also uniform along the surface in the basic state) \( \theta \) is constant (\( = \theta_s \), say) on the surface. Since the kinematic boundary condition is that the fluid on the surface must stay on the surface, \( \partial \theta = 0 \) there always. Now, (see Fig. 8) the actual surface is not at \( z = 0 \) (since that is our surface of \( p = p_0 \)) but at \( \phi = 0 \). So (for small amplitude perturbations), since \( \theta = \theta_s \) at the surface \( \phi = 0 \), then on \( z = 0, \theta = \theta_s - \delta z \partial \theta_0/\partial z \), so

\[
\theta' = -\delta z \left( \frac{\partial \theta_0}{\partial z} \right)
\]
there, where $\delta z$ is the (log $p$) height of the surface above $z = 0$. But on $z = 0$, $\phi' = -g \, \delta z$ and

$$\frac{\partial \phi'}{\partial z} = \frac{\kappa}{H} \bar{\Pi}(z) \, \phi',$$

so our boundary condition on $z = 0$ becomes

$$\frac{\partial \phi'}{\partial z} - \frac{N^2}{g} \phi' = 0.$$

Of all the possible values of $\mu$, then, only one satisfies the lower boundary condition: that for which $\mu = \mu_e$, where

$$\mu_e = \frac{N^2}{g}.$$

In order to satisfy the upper boundary condition, we must have $\mu = N^2/g < 1/2H$. For typical values, $N^2/g \approx 2 \times 10^{-5} m^{-1}$, whereas $1/2H \approx 6 \times 10^{-5} m^{-1}$, so there is no problem in practice. Note that this value of $\mu$ implies that the disturbance amplitude grows with height as $Z \sim \exp(\mu z)$, where $\mu^{-1} = 50 km$. (In the classic case of an isothermal atmosphere, $N^2/g = \kappa/H \approx (25 km)^{-1}$.) So the growth with height is fairly slow. This mode, which has no internal nodes and no phase change with height, is known as the external mode: it is in fact the only modal solution for $u_0 = 0$ and constant $N^2$: such an atmosphere has no internal modes\(^8\). In general, internal modes can be found only if $u_0$ and/or $N^2$ vary with height. Note that the external mode, which is analogous to surface waves on water (pressure variations at the surface playing the role of height variations of water) is not strictly barotropic (there are variations with $z$) but is sometimes referred to as equivalent barotropic.

\(^8\)This contrasts with the situation in the ocean which, because the ocean has discrete boundaries at top and bottom, possesses a full set of internal modes.
Now, note that the separation constant in (26) is now defined:

$$\frac{N^2}{g h_e} = -\mu \left( \mu - \frac{1}{H} \right) = \frac{N^2}{g H} \left( 1 - \frac{N^2 H}{g} \right).$$

The vertical structure problem thus determines $h_e$. Then, from (26), the $(x, y, t)$ structure of the mode satisfies

$$\left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \left\{ \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{f_0^2}{gh_e} \right] \psi' \right\} + \beta \frac{\partial \psi'}{\partial x} = 0,$$

which is exactly the same as the governing equation for a shallow water system of depth $h_e$, where

$$h_e = H \left( 1 - \frac{N^2 H}{g} \right)^{-1}. \quad (27)$$

Thus, $h_e$ is known as the equivalent depth for this mode: the mode’s structure (in $x, y,$ and $t$) is exactly the same as for a shallow water fluid of depth $h_e$, with the same mean flow $u_0$ and QGPV gradient $\beta$. For our values, $h_e \simeq 10$km. This, implies, e.g., that the deformation radius for this mode is $L_R = \sqrt{gh_e}/f_0 \simeq 3000$km.

(Note that, for an isothermal atmosphere, $N^2 = \kappa g/H$, so

$$h_e = H (1 - \kappa)^{-1} = \gamma H$$

where $\gamma = 1.4$ for air is the ratio of specific heats.)

### 3.6.2 Vertically propagating waves

For non-modal motions, we can look for vertically propagating solutions to (25), i.e. for real solutions for $m$ to

$$m^2 = \frac{N^2}{f_0^2} \left( \frac{\beta}{u_0 - c} - k^2 - l^2 \right) = \frac{1}{4H^2} .$$

Real $m$ is found for $0 < u_0 - c < U_c$, where the “Rossby critical velocity” is

$$U_c = \beta \left( k^2 + l^2 + \frac{f_0^2}{4N^2 H^2} \right)^{-1}.$$
$U_c$ is a function of wavenumber; for the very longest waves (zonal wavenumber $1, l = 0$) $U_c \simeq 100\text{ms}^{-1}$ and so there is a wide “window” of propagation, provided only that the winds are westerly and slower than $100\text{ms}^{-1}$. However, as wavenumber increases, the window closes. For zonal wave 4, $U_c \simeq 20\text{ms}^{-1}$; for zonal wave 10, $U_c \simeq 4\text{ms}^{-1}$. So only the planetary scale waves (scales larger than synoptic) have any real chance of propagating deeply in the vertical, and only then when the background flow is westerly.

Recall that

$$\psi' (x, y, z, t) = \exp \left( \frac{z}{2H} \right) \text{Re} \{ \Psi_0 \exp [i (kx + ly + mz - kct)] \} ,$$

so whenever propagation is allowed (real $m$), the wave amplitude grows with height as $e^{-z/2H}$, giving a factor of 10 amplification at 10hPa over the amplitude at the surface (if dissipation is negligible).

The impact of this selective propagation is most clearly seen in the stratosphere, where (Fig. 9) the zonal winds are easterly in summer (above 50-100 hPa) and strongly westerly in winter. As can be seen in Fig. 3.6.2, (for 30 hPa on 10 Jan 2006) wave disturbances are very weak in the summer easterlies, but strong and of planetary scale (in the case shown, wave 1 is dominant) in the winter westerlies. There is little if any source of waves in the stratosphere, so any waves there must propagate up from below. In summer, no waves can propagate; in winter, the long, planetary scale, waves can, but the synoptic scale disturbances, so prevalent in the troposphere, cannot.

### 3.7 Forced stationary waves

**Vertical structure**

Fig. 10 shows a longitude-height cross-section at 45°N of the response (perturbation streamfunction) to flow over a very localized mountain (location shown by arrow), in a zonal flow typical of northern mid-winter. The response splits into two wave trains: one is the external mode, propagating away from the forcing along the surface, the other is an upward propagating wave, which grows strongly in amplitude as it reaches the middle and upper stratosphere. These planetary Rossby waves dominate the meteorology of the winter stratosphere.

So the far-field response in the vertical can be seen simply as a vertically propagating Rossby wave; horizontal propagation within the troposphere is dominated by the external mode.
Figure 9: Zonal mean wind climatology for January
lon: plotted from 0.00 to 360
lat plotted from 20.00 to 90.00
lev: 30.00
T: Jan 10 2006 00 Z
Individual obs hgt m
3.7.1 External mode response

We saw that the external mode is described by the shallow water equations, so we will use these. Before doing so, let us recall the observed structure of tropospheric stationary waves (Fig. 11). The absence of a vertical phase tilt is consistent with an interpretation that these waves are dominated by the external mode.

We want to consider the response of the atmosphere, with a westerly prevailing flow, to stationary forcing (so $c = 0$). We found the dispersion relation

$$\omega = u_0 k - \frac{\beta k}{k^2 + l^2 + L_R^{-2}}. \quad (28)$$

where now $L_R = \sqrt{gh_e}/f_0$. Anticipating a stationary response to stationary

Figure 10: Perturbation streamfunction as a function of longitude and height for a 3D calculation of the response to flow over an isolated mountain (location marked by an arrow). [From Held, 1983.]
forcing, we require $\omega = 0$, which requires

$$k^2 + l^2 = \kappa_s^2 - L_R^2,$$

where $\kappa_s = \sqrt{\beta/u_0}$ is the stationary wavenumber. For $L_R = 3 \times 10^6$ m, and typical midlatitude values $u_0 = 30$ m s$^{-1}$, $\beta = 1.6 \times 10^{-11}$ m$^{-1}$ s$^{-1}$, $\kappa_s^{-1} \approx 1500$ km, so such waves have typical wavelength $2\pi/\kappa_s \approx 9000$ km, which at $45^\circ$ latitude corresponds approximately to zonal wavenumber 3.

Now, from (28), the zonal component of group velocity is

$$c_{gx} = u_0 + \beta \frac{(k^2 - l^2 - L_R^{-2})}{(k^2 + l^2 + L_R^{-2})^2};$$

given (29) and some manipulation, it follows that, for our stationary waves with $k^2 + l^2 + L_R^{-2} = \beta/u_0$,

$$c_{gx}(\omega = 0) = 2k^2 \frac{u_0^2}{\beta};$$

the zonal component of group velocity is eastward. (The north-south component can be in either direction.) Even though, in general, Rossby waves can have eastward or westward group velocity, depending on scale, stationary waves are of a specific scale and must propagate (in a group velocity sense)
Figure 12: Flow over a localized mountain. Numerical solutions for the perturbation streamfunction $\psi'$ for flow over (left) mountains in the eastern hemisphere (Tibet, mostly, with a small contribution from the Alps) and (right) the western hemisphere (mostly the Rockies). Note the Rossby waves propagating “downstream” (eastward) of the mountains. [Held 1983]
eastward: so the response to localized forcing will be a wavetrain located to the east of the forcing.

These characteristics are evident in Fig. 12, which shows the response to northern hemisphere topographic forcing in a model with realistic northern hemisphere winter flow, where the topography has been separated into western and eastern hemispheres (the former being dominated by the Rockies, the latter by the Himalaya, with a smaller contribution from the Alps).

### 3.8 Wave refraction on the sphere

On the sphere, the inviscid eqs. of motion (for a shallow water system) become

\[
\frac{du}{dt} - fv = \frac{\partial u}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{a \varphi} \frac{\partial u}{\partial \varphi} - \frac{uv}{a} \tan \varphi -fv = \frac{g}{a \cos \varphi} \frac{\partial h}{\partial \lambda},
\]

\[
\frac{dv}{dt} + fu = \frac{\partial v}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{v}{a \varphi} \frac{\partial v}{\partial \varphi} + \frac{v^2}{a} \tan \varphi + fu = -\frac{g}{a \varphi} \frac{\partial h}{\partial \varphi},
\]

where \( f = 2\Omega \sin \varphi \) and \( a \) is the Earth radius, so the geostrophic relationships are just

\[
u = -\frac{1}{a} \frac{\partial \psi}{\partial \varphi}; \quad v = \frac{1}{a \cos \varphi} \frac{\partial \psi}{\partial \lambda},
\]

(30)

where the geostrophic streamfunction is

\[
\psi = \frac{gh}{f},
\]

as before. The curl operator on a sphere gives vorticity

\[
\zeta = \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi)
\]

\[
= \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 \psi}{\partial \lambda^2} + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left( \cos \varphi \frac{\partial \psi}{\partial \varphi} \right),
\]

(31)

Taking the curl of the equations of motion once again gives the inviscid shallow water vorticity equation

\[
\frac{d\Pi}{dt} = \frac{\partial \Pi}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial \Pi}{\partial \lambda} + \frac{v}{a \varphi} \frac{\partial \Pi}{\partial \varphi} = 0,
\]
where \( \Pi = (f + \zeta) / h \), as before. As usual, we look at small amplitude perturbations to a zonal basic state: we allow the background flow \( U(\phi) \) to vary with latitude, with a corresponding surface height \( H(\phi) \) in geostrophic balance, such that

\[
U = -\frac{g}{af} \frac{\partial H}{\partial \phi}.
\]

The basic state has PV (using (31))

\[
\Pi_0 = \left[ f - \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (U \cos \phi) \right] / H.
\]

Linearizing the perturbation PV equation then gives

\[
\frac{\partial \Pi'}{\partial t} + \frac{U}{a \cos \phi} \frac{\partial \Pi'}{\partial \lambda} + v' \frac{\tilde{\beta}}{H} = 0,
\]

(32)

where

\[
\tilde{\beta}(\phi) = \frac{H}{a} \frac{d\Pi_0}{d\phi}
\]

is proportional to the basic state PV gradient (note that \( \tilde{\beta} = \beta \) if \( U = 0 \) and \( H \) is constant). Since

\[
\Pi' = \frac{\zeta'}{H} - \frac{f}{H^2} h'
\]

(where we have assumed \( |\zeta| \ll |f| \) and \( |h'| \ll H \), on the assumption of small \( Ro \)) we therefore have

\[
\left( \frac{\partial}{\partial t} + \frac{U}{a \cos \phi} \frac{\partial}{\partial \lambda} \right) \left[ \frac{1}{a^2 \cos^2 \phi} \frac{\partial^2 \psi'}{\partial \lambda^2} + \frac{1}{a^2 \cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial \psi'}{\partial \phi} \right) - \frac{1}{\lambda^2} \psi' \right] + \frac{\tilde{\beta}}{a \cos \phi} \frac{\partial \psi'}{\partial \lambda} = 0.
\]

Now, if we look for solutions that are wavelike in \( \lambda \) and \( t \),

\[
\psi'(\lambda, \phi, t) = \text{Re} \Psi(\phi) \exp \left[ is (\lambda - \sigma t) \right],
\]

where \( s \) is the zonal wavenumber and \( \sigma \) the phase speed, both in longitude units, we have

\[
-is \left( \sigma - \frac{U}{a \cos \phi} \right) \left[ -\frac{s^2}{a^2 \cos^2 \phi} \Psi + \frac{1}{a^2 \cos \phi} \frac{d}{d\phi} \left( \cos \phi \frac{d\Psi}{d\phi} \right) - \frac{1}{\lambda^2} \psi' \right] + \frac{is \tilde{\beta}}{a \cos \phi} \Psi = 0.
\]
or
\[
\frac{1}{a^2 \cos \varphi} \frac{d}{d\varphi} \left( \cos \varphi \frac{d\Psi}{d\varphi} \right) + \left[ \frac{\tilde{\beta}}{(U - c)} - \frac{s^2}{a^2 \cos^2 \varphi} - \frac{1}{L_R^2} \right] \Psi = 0
\]
where \( c = a \sigma \cos \varphi \) is the phase speed in distance units. This equation can be written
\[
\frac{1}{a^2 \cos \varphi} \frac{d}{d\varphi} \left( \cos \varphi \frac{d\Psi}{d\varphi} \right) + \nu^2 \Psi = 0 , \quad (33)
\]
where
\[
\nu^2 = \frac{\tilde{\beta}}{(U - c)} - \frac{s^2}{a^2 \cos^2 \varphi} - \frac{1}{L_R^2} \quad (34)
\]
is the square of the “refractive index” for equivalent barotropic Rossby waves on the sphere.

Now, for waves whose length scale \( \ll a \), and for a basic state that is slowly varying in the sense that allows us to use the WKB approximation, then since \( a \, d\varphi = dy \), (33) gives
\[
\Psi \sim e^{\pm i\nu y}
\]
so \( \nu \) is just the local (dimensional) wavenumber in latitude. Now, waves tend to refract toward larger refractive index. For the case of no mean flow, or solid body rotation \( (U = \omega a \cos \varphi, \) where \( \omega \) is a constant angular velocity) then since \( c = \sigma a \cos \varphi \), and \( \tilde{\beta} = \tilde{\beta} = 2 \Omega a^{-1} \cos \varphi \), the first and last terms in (34) are constant, and so only the second term varies with \( \varphi \). This term \( \sim \cos^{-2} \varphi \)
and thus always becomes more negative poleward; so if \( \nu^2 \) is positive (which it must be, if we have propagating waves) it increases equatorward. (In fact, \( \nu^2 \) must become negative sufficiently close to the pole.) This effect, a purely geometric one which has nothing to do with the structure of the basic state, means the waves appear to refract equatorward, as we saw in the Rossby wave solutions to localized forcing on the sphere. (“Appear” because in fact wave rays propagate along great circles in this case.) In the presence of realistic winds, this effect is in fact enhanced, since \( U - c \) decreases equatorward as midlatitude westerlies give way to tropical easterlies. In fact, (33) becomes singular where \( U - c \to 0 \), and we have to consider what happens there from a different perspective.

### 3.9 Quasi-linear waves near the critical line

So what does happen at the “critical line,” where \( U - c = 0 \), and the linear problem becomes singular? In fact, the assumptions on which our lin-
earization was based (e.g., that we can neglect \( u' \partial q'/\partial x \) compared with \( (U - c) \partial q'/\partial x \)) break down there. Nevertheless, we can grasp a basic understanding of what goes on by basing a discussion on linear theory—as long as we are careful. Consider small-amplitude, adiabatic stationary planetary waves in a shear flow \( U(y) \) which contains a critical line \( U = c \). To make things tidier, ignore spherical geometry (its effects do not matter close to the critical line) when our perturbation PV equation (32) becomes

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \Pi' + \frac{\tilde{\beta}}{H} v' = 0 .
\]

Let the northward displacement (in a Lagrangian sense) of an air parcel be \( \eta' \) such that

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \eta' \equiv v' .
\]

Then, if \( \Pi' = 0 \) when \( \eta' = 0 \) (and we can choose to define \( \eta' \) in this way), we have

\[
\Pi' = -\eta' \frac{\tilde{\beta}}{H} .
\]

(We could have got this from a Taylor expansion about the mean position of a \( \Pi \)-contour). Now, since \( v' = \text{Re} \, e^{ik} \Psi \, e^{ik(x - ct)} \) then we have \( U \partial \eta'/\partial x = v' \), or

\[
\eta' = \text{Re} \frac{\Psi}{U - c} e^{ik(x - ct)} ,
\]

which is singular where \( U = c \). The reasons for this singularity are not hard to find. Consider the initial value case where \( \eta' = 0 \) at \( t = 0 \). Then if

\[
\eta' = \text{Re} \, Y(y, t)e^{ik(x - ct)} ,
\]

\[
\left[ \frac{\partial}{\partial t} + ik(U - c) \right] Y = i k \Psi ,
\]

whence

\[
Y = \frac{\Psi}{(U - c)} (1 - e^{-ik(U-c)t}) .
\]

For \( U - c \) finite, this means that \( Y \) just oscillates around the steady solution \( Y_0 = \Psi / (U - c) \) and the dynamics are wavelike, as shown in Fig 13. For \( k(U - c) t \ll 1 \), however (and this will be true for an increasingly long time as \( U - c \to 0 \)),

\[
Y \approx i k t \, \Psi
\]
—so $\eta'$ just grows linearly with time (i.e., $\partial \eta'/\partial t = v'$). There is therefore (see Fig 13) no oscillation at the critical line; the parcel displacements just increase systematically with time. This is more characteristic of a chaotic flow than a wavelike one and, indeed, is indicative of the presence of closed eddies (see Fig 14, below)—rather than waves on a prevailing westerly flow—near the critical line. Note that $\Pi'$ will behave in the same way as $\eta'$. Note that, because of the monotonic growth of $\eta'$ at the critical line, linear theory must eventually break down there.

3.10 The finite-amplitude critical layer

Clearly, given enough time, our linear assumption will break down at the critical line (and our assumption of a steady linear wave is never valid there). The total streamfunction is

$$\psi = \psi_0 + \psi'$$

$$= - \int U(y) \, dy + \text{Re} \Psi(y) e^{ik(x-ct)} .$$

As shown in Fig. 14, the streamlines do not form continuous, wavy, lines near the critical line (as they do elsewhere) but form closed, anticyclonic, eddies within a region known as the critical layer. Since PV is conserved, it is advected along the streamlines: outside the critical layer, the PV contours just align themselves with the streamlines. Within the critical layer, however,
Figure 14: Streamlines in the vicinity of the critical layer in a linear shear $U = \Lambda y$: the total streamfunction is then $\psi = -\Lambda y^2 + \Psi \cos kx$. This figure is drawn in the frame of reference of the propagating wave (i.e., in which $c = 0$). Dashed contour encloses the “Kelvin cat’s eyes” region of closed eddies. The undisturbed zero wind line where $U = c$ is at $y = 0$.

y they get wrapped around the eddies, as shown in Fig. 15. thus, the PV (along with any other tracer) gets stirred and mixed within the critical layer. We’ll see ways in which this is important later; for our immediate purposes, note that this implies that the background PV gradient, on which the wave “rides,” is being destroyed: the wave is breaking. By cascading PV down to small scales, the wave is thus dissipating itself. This is the primary reason why waves on the sphere do not propagate across the equator into the other hemisphere.
Figure 15: Evolution of PV in a critical layer (after Haynes, *J. Fluid Mech.*, 161, 493-511 [1985]).