5 Waves in the tropical atmosphere

5.1 Tropical dynamics: equatorial waves

[Holton, Ch.11.4; Gill]

We will now focus on the theory of tropical dynamics. Even though moist convection plays a central role here, we will concentrate on the “dry” fluid dynamics of the region, relegating moisture effects to the status of externals. (This can be a dangerous game if not done carefully, but to do otherwise would lead us too far outside the scope of this class.)

We return to the shallow water system. Compared with the midlatitude case, we cannot assume QG dynamics (because Rossby number $U/fL$ may not be small in the tropics where $f$ is small and we can no longer sustain the assumption $\beta L/f_0 \ll 1$). Nevertheless, we can still assume $f$ is a linear function of $y$ (it is in fact a very good approximation close to the equator, where $d^2f/dy^2 = 0$) but now

$$f(y) = \beta y$$

where $y$ is the distance from the equator, and $\beta = 2\Omega/a$, provided we make no assumption about $\beta L$ being small. This leads us to the equatorial beta-plane, just like our midlatitude beta-plane, but centered on the equator with $f_0 = 0$.

The shallow water equations now become

$$\frac{du}{dt} - \beta yv = -g \frac{\partial h}{\partial x}$$
$$\frac{dv}{dt} + \beta yu = -g \frac{\partial h}{\partial y}$$
$$\frac{dh}{dt} + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0.$$ (1)

The mathematical complication, compared with the midlatitude case, is that the coefficient of $u$ and $v$ in the Coriolis terms is now a function of $y$ at leading order.

As usual, we look at the behavior of small amplitude perturbations to a resting basic state, with uniform depth $D$. The linearized perturbation
The equations become

\[
\begin{align*}
\frac{\partial u'}{\partial t} - \beta y v' &= -g \frac{\partial h'}{\partial x} \\
\frac{\partial v'}{\partial t} + \beta y u' &= -g \frac{\partial h'}{\partial y} \\
\frac{\partial h'}{\partial t} + D \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= 0
\end{align*}
\] (2)

The coefficients are independent of \( x \) and \( t \) (but not \( y \)) so we look for solutions of the form

\[
\begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} = \text{Re} \left( \begin{pmatrix} u_0(y) \\ v_0(y) \\ h_0(y) \end{pmatrix} e^{i(kx-\omega t)} \right),
\]

giving

\[
\begin{align*}
-i\omega u_0 + \beta y v_0 &= -ikgh_0 \\
-i\omega v_0 + \beta y u_0 &= -g \frac{dh_0}{dy} \\
-i\omega h_0 + D \left( \frac{dv_0}{dy} + iku_0 \right) &= 0
\end{align*}
\]

Eliminating \( u_0 \) from the first two gives

\[
(\beta^2 y^2 - \omega^2) v_0 - ig \left( k\beta y h_0 + \omega \frac{dh_0}{dy} \right) = 0 ,
\]

and from the first and third gives

\[
(\omega^2 - gDk^2) h_0 + i\omega D \left( \frac{dv_0}{dy} - \frac{k}{\omega} \beta y v_0 \right) = 0 .
\]

Elimination of \( h_0 \) between these (and assuming \( \omega^2 \neq gDk^2 \)) gives

\[
\frac{d^2 v_0}{dy^2} + \left[ \left( \frac{\omega^2}{gD} - k^2 - \frac{k}{\omega} \beta \right) - \frac{\beta^2 y^2}{gD} \right] v_0 = 0 .
\] (3)

This equation cleans up a little if we introduce the dimensionless distance\(^1\)

\[
\xi = \frac{y}{L_E}
\]

\(^1\)Note that we cannot do this for the barotropic case \((D \to \infty)\); we’ll discuss the barotropic case below.
where
\[ L_E = (gD)^{1/4} \beta^{-1/2} \quad (4) \]
is the equatorial radius of deformation. [Recall that, in middle latitudes, the deformation radius was defined as \( L_R = (gD)^{1/2} f^{-1} \). But a distance \( L_R \) from the equator, \( f = \beta L_R \), so
\[ L_R = \frac{(gD)^{1/2}}{\beta L_R} , \]
whence the definition (4) at the equator.]

With this substitution, (3) becomes
\[ \frac{d^2v_0}{d\xi^2} + (\Gamma - \xi^2) v_0 = 0 , \quad (5) \]
where
\[ \Gamma = \frac{\sqrt{gD}}{\beta} \left( \frac{\omega^2}{gD} - k^2 - \frac{k}{\omega} \beta \right) . \quad (6) \]

(5) is a well known equation (the quantum harmonic oscillator). Together with the boundary conditions that \( v_0 \) is bounded as \( y \to \pm \infty \), it represents an eigenvalue problem for \( \Gamma \) (and hence for \( \omega \)), which has eigenvalues
\[ \frac{\sqrt{gD}}{\beta} \left( \frac{\omega_n^2}{gD} - k^2 - \frac{k}{\omega_n} \beta \right) = \Gamma_n = 2n + 1 , \ n \geq 0 . \quad (7) \]
The corresponding eigenvectors are Hermite functions \( V_n(\xi) \) [so the solutions to (5) are \( v_0 = \text{constant} \times V_n(\xi) \).]:
\[ V_n(\xi) = \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(\xi) \exp \left( -\frac{1}{2} \xi^2 \right) , \quad (8) \]
where \( H_n(\xi) \) are Hermite polynomials, defined by
\[ H_n(\xi) = (-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2) . \quad (9) \]
The first few polynomials are
\[ H_0 = 1 , \quad H_1 = 2\xi , \quad H_2 = 4\xi^2 - 2 , \ldots . \]
The first 3 functions \( H_n(\xi)e^{-\frac{1}{2}x^2} \) are plotted in Fig. 1. The solutions are
trapped within a distance $\xi = O(1) [y, L_E]$ of the equator: there is an equatorial waveguide. The $n^{th}$ mode has $n$ nodes in $y$.

The dispersion relation is given by the roots of (7), which can be rewritten

$$\omega_n^2 - \frac{k'^2}{\omega'} - \frac{k'}{\omega'} - 2n - 1 = 0$$

where $k' = kL_E$ and $\omega' = \omega L_e/\sqrt{gD}$. $\omega'(k')$ is plotted in Fig. 2. The roots fall into 3 branches, most clearly separated as $k' \to \infty$. In this limit, (10) gives either $\omega' = \pm k'$ or $\omega' = -k'$, or, in dimensional terms,

$$\omega \to \begin{cases} 
\pm \sqrt{gD} \\
-\beta/k'
\end{cases}, \quad kL_E \to \infty .$$

The first pair of these we recognize as gravity waves, the third as Rossby waves in the large $k$ limit.

The structure of the two gravest modes (the $n = 1$ Rossby wave, and the $n = 0$ [Yanai, or mixed Rossby-gravity, wave]) are shown in Fig. 3.

5.1.1 The long wave, low frequency limit
For small $k$, and small $\omega$, (10) takes the limiting form

$$-\frac{k'}{\omega'} - 2n - 1 = 0$$
Figure 2: The (dimensionless) equatorial wave dispersion relation.
Figure 3: Structure ($h$, contours, and $u$, arrows) for the Kelvin, Yanai, and gravest Rossby wave with $k' = 1$. 
whence
\[ \omega' = -\frac{k'}{2n+1}. \]  
This is the limiting form of the Rossby wave dispersion relation; in this limit the waves become non-dispersive. Note that this limiting solution is valid only for \( n \geq 1 \) (see Section 5.1.2).

5.1.2 The case \( n = 0 \)

Note from Fig. 2 that there are only two branches for \( n = 0 \), even though (10) remains a cubic for \( n = 0 \). Why? In this case, (10) can be written
\[ (\omega' + k') \left( k' + \frac{1}{\omega'} - \omega' \right) = 0, \]
so that one of the roots is, exactly, \( \omega' = -k' \). In dimensional terms, this is \( \omega = -gDk \), and the root is thus spurious, since this value of \( \omega \) renders our derivation of (3) invalid. So there is no \( n = 0 \) member of the westward propagating gravity wave branch. However, the root for \( n = 0 \) that falls on the Rossby wave branch for large \( k \) crosses over to join the gravity wave branch at small \( k \). For this reason, this mode is known as the mixed Rossby-gravity wave. Its structure is shown in Fig. 3.

5.1.3 The equatorial Kelvin wave (\( n = -1 \))

There is in fact another solution to (3) that we have not yet covered: (3) is solved trivially by \( v_0 = 0 \). It is not usual for such a trivial solution to be valid, or interesting, but in this case it is both. In fact, the derivation of (3) is problematic in this case, but if we return to our basic equations (2) and set \( v = 0 \) everywhere, we get
\[
\frac{\partial u'}{\partial t} = -g \frac{\partial h'}{\partial x}, \\
\beta y u' = -g \frac{\partial h'}{\partial y}, \\
\frac{\partial h'}{\partial t} + D \frac{\partial u'}{\partial x} = 0 \tag{12}
\]
Note that the \( y \)-component of the momentum eqs. gives geostrophic balance (despite being near the equator). The first and third of (12) are in fact
identical to the equations for nonrotating gravity waves, giving

\[-i\omega u_0 = -ikgh_0 \; ; \; -i\omega h_0 + ikDu_0 = 0\]

whence

\[c = \frac{\omega}{k} = \pm \sqrt{gD}, \quad (13)\]

as we would expect. However, only one of these apparent solutions is valid. The second of (12) together with the first gives us

\[-g \frac{dh_0}{dy} = \beta yu_0 = \frac{g}{c} \beta yh_0 ,\]

whence

\[h_0(y) = \text{constant} \times \exp \left( -\frac{\beta}{2c} y^2 \right).\]

But the solution must be bounded as \(|y| \to \infty\), so \(c\) must be positive: hence the Kelvin wave has phase speed

\[c_K = + \sqrt{gD}\]

— it propagates eastward. It is also nondispersive, and has the latitudinal structure

\[h_K(y) = \text{constant} \times \exp \left[ -\frac{1}{2} \left( \frac{y}{L_E} \right)^2 \right] = \text{constant} \times V_0 (\xi).\]

Its structure is shown in Fig. 3.

5.1.4 The barotropic case \((L_E \to \infty)\)

In the barotropic limit, the equatorial waveguide becomes infinitely wide, so the treatment becomes problematic. In fact, this limit is very simple. Go back to (3) and set \((gD)^{-1} = 0:\)

\[\frac{d^2 v_0}{dy^2} - \left( k^2 + \frac{k}{\omega} \beta \right) v_0 = 0.\]

This equation now has constant coefficients and has solutions \(v_0 \sim \exp (\pm ily)\) with

\[\omega = -\frac{\beta k}{k^2 + l^2},\]
i.e., plane waves, with the same Rossby wave dispersion relation as we had for the midlatitude beta-plane (in the barotropic limit). There is no equatorial waveguide. This follows simply from the fact that the barotropic vorticity budget cares about \( \beta \), but not \( f_0 \) (in the general case, \( f_0 \) appears in the stretching term, which is of course absent in the barotropic limit). So the equator has no special role in the barotropic problem, and so tropical Rossby waves are no different from midlatitude ones.

5.2 Observed equatorial waves

A very revealing analysis was done by Wheeler & Kiladis [ *J. Atmos. Sci.*, **56**, 379-399 (1999)], who looked at the space-time spectrum of fluctuations evident in outgoing longwave radiation (OLR) data. The OLR signal is essentially a measure of cloud-top temperature of deep convective systems and hence what is being observed are *convectively coupled waves* — i.e. those waves that are coupled with, and made visible by, deep convection. Their results are summarized in Fig. 4. Several clear bands are evident. The power spectrum of the antisymmetric components shows a band that is well fitted by the dispersion relation of a mixed Rossby-gravity wave \((n = 0)\) with equivalent depth \(25\) m. That of the symmetric components shows several features: an eastward-propagating, apparent Kelvin wave band with \(h_e \approx 25 - 50\) m, hints of westward-propagating Rossby wave components, and a slow eastward-propagating “Madden-Julian Oscillation.”

Why an apparent \(25\) m (or so) equivalent depth? The coupling with deep (and moist) convection implies a weak effective stability, and the vertical structure of most large-scale tropical motions is “first-baroclinic-mode-like” in that variables like horizontal velocity have one sign in the lower troposphere, and the opposite in the upper troposphere, with implied vertical velocity single-signed, with a mid-tropospheric peak. But why should it be like that? Despite what instinct might suggest, the high static stability of the stratosphere does not act as a lid on the troposphere: rather, it encourages wave propagation above the tropopause and so does not permit first-baroclinic-mode structures. Current thinking seems to be that this kind of vertical structure is not really mode-like, but is imposed on the dynamics by the vertical structure of deep convection.

Just like extratropical Rossby waves, equatorial waves can also adopt both external-mode-like (which are very fast, \(c \sim 300\) ms\(^{-1}\)) and vertically propagating forms; upward propagating equatorial waves are observed in the
Figure 4: OLR power spectra, from Wheeler & Kiladis. Spectra have been scaled by the “background” values to enhance the signal. Data were separated into components antisymmetric (left) and symmetric (right) about the equator. The solid curves are theoretical shallow water dispersion relations with equivalent depths of 12, 25, and 50m.
stratosphere, where they seem to play a significant role in tropical dynamics.

5.3 Steady, zonally asymmetric circulations in the tropics: the Gill model


We consider the zonally varying circulation forced by localized steady thermal forcing (such as that associated with the region of intense deep convection over the west equatorial Pacific Ocean) on the equatorial \( \beta \)-plane. In order to use a shallow water model, we need to determine an equivalent depth for baroclinic motions. We will do this by taking a fit to observed convectively-coupled wave characteristics; in fact the whole approach assumes a “first baroclinic mode” vertical structure, the existence of which (wrongly) requires a lid on the troposphere. We’ll discuss the appropriateness of this later.

We linearize about a stably stratified state of no motion. The perturbation thermodynamic equation is

\[
\frac{\partial T'}{\partial t} + w'S = \frac{J}{\rho c_p},
\]

where \( S = \left( \frac{dT_0}{dz} + \frac{g}{c_p} \right) \). To relate this to a shallow water model of depth \( D \), we replace

\[
T' \rightarrow -h'S, \\
w' \rightarrow -D \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right), \\
J \rightarrow D \rho c_p S Q
\]

to give a set of perturbation shallow water equations

\[
\begin{align*}
\frac{\partial u'}{\partial t} - \beta y v' &= -g \frac{\partial h'}{\partial x} \\
\frac{\partial v'}{\partial t} + \beta u' &= -g \frac{\partial h'}{\partial y} \\
\frac{\partial h'}{\partial t} + D \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= -DQ
\end{align*}
\]

which are the same as the unforced set but with the addition of the “thermal” forcing \( Q \). In order to incorporate dissipation (we’ll see that this is important
to the solutions) we include both linear drag and “Newtonian” cooling, each with the same rate \( \ddot{\varepsilon} \), to get

\[
\begin{align*}
\frac{\partial u'}{\partial t} - \beta yv' &= -g \frac{\partial h'}{\partial x} - \ddot{\varepsilon} u' \\
\frac{\partial v'}{\partial t} + \beta yu' &= -g \frac{\partial h'}{\partial y} - \ddot{\varepsilon} v' \\
\frac{\partial h'}{\partial t} + D \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= -DQ - \ddot{\varepsilon} h'
\end{align*}
\]

Now we non-dimensionalize\(^2\), using the deformation radius \( L_E = (c/\beta)^{1/2} \), and the time scale \( \tau_E = L_E/c = (c\beta)^{-1/2} \), where \( c = (gD)^{1/2} \) is the gravity wave speed. If we choose, empirically (on the basis of observed speeds of convecting equatorial waves), an equivalent depth \( D = 30 \) m, then \( c = \sqrt{9.81 \times 30} = 17 \) ms\(^{-1}\) and, with \( \beta = 2.28 \times 10^{-11}\text{m}^{-1}\text{s}^{-1} \), we have \( L_E = 860 \) km, and \( \tau_E = 50800 \) s \( \simeq 14 \) hr. Then, with \( u' = cu, \ v' = cv, \ y \rightarrow L_E y, \ x \rightarrow L_E x, \ h' = Dh \), we get

\[
\begin{align*}
\frac{\partial u}{\partial t} - yv &= -\frac{\partial h}{\partial x} - \varepsilon u \\
\frac{\partial v}{\partial t} + yu &= -\frac{\partial h}{\partial y} - \varepsilon v \\
\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -Q - \varepsilon h
\end{align*}
\]

where \( \varepsilon = \tau_E \ddot{\varepsilon} \). Finally (motivated by the observation that the zonal scale of the Walker circulation is around \( 10^4 \) km) we make the “long wave” approximation that the zonal length scale \( \gg L_E \); this implies that \( |u| \gg |v| \) and so, provided \( \varepsilon \lesssim 1 \), we can neglect \( \varepsilon v \) in the second equation. Then, in steady state, we have

\[
\begin{align*}
\varepsilon u - yv &= -\frac{\partial h}{\partial x} \\
yu &= -\frac{\partial h}{\partial y} \\
\varepsilon h + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -Q .
\end{align*}
\]
Now, if we associate $\varepsilon$ with $-i\omega$, we know that the homogeneous (unforced) version of (14) have eigenfunctions that are Hermite functions, so we anticipate that expanding in these functions will be a good approach to the problem. First, however, we follow Gill by writing

$$u = \frac{1}{2} (q - r)$$

$$h = \frac{1}{2} (q + r)$$

when (14) can be rewritten and reorganized to

$$\varepsilon q + \frac{\partial q}{\partial x} + \frac{\partial v}{\partial y} - yv = -Q$$

$$\varepsilon r - \frac{\partial r}{\partial x} + \frac{\partial v}{\partial y} + yv = -Q$$

$$\frac{\partial q}{\partial y} + yq + \frac{\partial r}{\partial y} - yr = 0.$$  \hspace{1cm} (15)

Now we expand

$$\begin{pmatrix} q \\ r \\ v \\ Q \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} q_n(x) \\ r_n(x) \\ v_n(x) \\ Q_n(x) \end{pmatrix} V_n(y)$$

where $V_n$ are the Hermite functions we encountered previously. They have the recurrence relations

$$\frac{dV_n}{dy} + yV_n = \sqrt{2n}V_{n-1}$$

$$\frac{dV_n}{dy} - yV_n = -\sqrt{2} (n + 1) V_{n+1}$$  \hspace{1cm} (17)

$$\sqrt{2} (n + 1) V_{n+1} - 2yV_n + \sqrt{2n} V_{n-1} = 0$$
Substituting into (15) and using (17), we get

\[
\sum_{n=0}^{\infty} \left[ (\varepsilon q_n + \frac{\partial q_n}{\partial x} + Q_n) V_n - \sqrt{2(n+1)}v_n V_{n+1} \right] = 0
\]

\[
\sum_{n=0}^{\infty} \left[ (\varepsilon r_n - \frac{\partial r_n}{\partial x} + Q_n) V_n + \sqrt{2n}v_n V_{n-1} \right] = 0
\]  \hspace{1cm} (18)

\[
\sum_{n=0}^{\infty} \left[ \sqrt{2n}q_n V_{n-1} - \sqrt{2(n+1)}r_n V_{n+1} \right] = 0
\]

Given the forcing \(Q(x, y) = \sum Q_n(x)V_n(y)\), equations (18) can be used to build the solution. Note that the Hermite functions are normalized such that

\[
\int_{-\infty}^{\infty} V_m(y) V_n(y) \, dy = \delta_{nm}, \hspace{1cm} (19)
\]

so that

\[
Q_n(x) = \int_{-\infty}^{\infty} Q(x, y) V_n(y) \, dy.
\]  \hspace{1cm} (20)

### 5.3.1 Localized forcing on the equator

Suppose that \(Q\) is localized in \(x\), being zero except in \(-L < x < L\), say, and that in latitude it is simply described by \(V_0(y) = \exp \left( -\frac{1}{2}y^2 \right)\), so that only \(Q_0(x)\) is nonzero. Then (18) gives us

\[
\varepsilon q_0 + \frac{\partial q_0}{\partial x} = -Q_0
\]

\[
\varepsilon q_2 + \frac{\partial q_2}{\partial x} = 0
\]

\[
\varepsilon r_0 - \frac{\partial r_0}{\partial x} + \sqrt{2}v_1 = -Q_0
\]

\[
2q_2 - \sqrt{2}r_0 = 0
\]

and all other coefficients are zero. We can use the 2nd, 3rd and 4th of these to get

\[
3\varepsilon q_2 - \frac{\partial q_2}{\partial x} = -\sqrt{2}Q_0. \hspace{1cm} (21)
\]

Taking the first equation first, then outside the forcing region,

\[
\varepsilon q_0 + \frac{\partial q_0}{\partial x} = 0
\]
Demanding a bounded solution leads us to conclude that

\[
q_0(x) = \begin{cases} 
0, & x < -L \\
-f_L^x Q_0(x') e^{(x'-x)} dx', & -L < x < 0 \\
-e^{\varepsilon x} \int_{-L}^x Q_0(x') e^{\varepsilon x'} dx', & x > L
\end{cases}
\]  

This part of the solution, which has \( u = h \sim \exp\left(-\frac{1}{2}y^2\right) \) and \( v = 0 \) is the Kelvin wave part. There is no Kelvin wave response to the west of the forcing simply because the Kelvin wave group velocity is eastward. To the east the solution decays as \( e^{-x/\lambda} \) where

\[
\lambda = \varepsilon^{-1} = \frac{\text{group velocity}}{\text{dissipation rate}}
\]

(recall that in dimensional terms \( \omega = k \) for the Kelvin wave, whence the dimensionless group velocity is unity).

The second part of the solution, from (21), has

\[
q_2(x) = \begin{cases} 
-\sqrt{2} e^{3x} \int_{-L}^x Q_0(x') e^{-3x'} dx', & x < -L \\
\sqrt{2} \int_{-L}^x Q_0(x') e^{3(x-x')} dx', & -L < x < L \\
0, & x > L
\end{cases}
\]  

Here boundedness demands that there is no component like \( e^{3x} \) in \( x > L \), to the east of the forcing. This part of the solution also includes \( v_1(x) \) and \( r_0(x) \), where

\[
2v_1 = \varepsilon q_2 + \frac{\partial q_2}{\partial x} \\
r_0 = \sqrt{2} q_2
\]

Note that this component is zero to the east of the forcing and, to the west, varies as \( e^{3x} \). This is the \( n = 1 \) Rossby wave component. Recall that, in the long wave limit under consideration here, the \( n = 1 \) Rossby wave has \( \omega = -k/3 \), so that its group velocity is \(-1/3\). Hence there is no contribution to the east of the forcing and to the west the solution decays as \( e^{x/\lambda} \), where now

\[
\lambda = (3\varepsilon)^{-1} = \frac{|\text{group velocity}|}{\text{dissipation rate}}
\]

Since the group velocity of the \( n = 1 \) Rossby wave is \( 1/3 \) of that of the Kelvin wave, it attenuates a factor of 3 more rapidly than the Kelvin wave.
All other components of the expansion (16) are zero, a consequence of the fact that we chose a forcing structure (in y) that does not project onto the higher modes. The full solution is shown in Fig. 5; the central frame, showing low-level pressure and wind, clearly shows the predicted characteristics: zonal flow to the east, with a Gaussian latitudinal profile of $u$ and $p$ ($\equiv h$), and twin cyclones straddling the equator to the west, as expected for the $n = 1$
Rossby wave, which has $h$ structure

$$
\begin{align*}
  h &= \frac{1}{2} (q - r) \\
  &= \frac{1}{2} \left[ q_2(x)V_2(y) + r_0(x)V_0(y) \right] \\
  &= \frac{1}{2} \left[ q_2(x) \left[ V_2(y) + \sqrt{2}V_0(y) \right] \right] \\
  &= \frac{1}{\sqrt{2\sqrt{\pi}}} q_2(x) (2y^2 + 1) e^{-y^2/2}.
\end{align*}
$$

This function has extrema in $y$ at $y = 0$ and $y = \pm \sqrt{3/2} = \pm 1.22$, consistent with the locations of the twin low level cyclones on the figure. [Note that Gill’s values for $x$ and $y$ must be divided by $\sqrt{2}$ to convert to our notation.] With our numbers, $L_E \approx 860 \text{ km}$, so these values of $y$ correspond to dimensional distance of $1050 \text{ km}$, or about $\pm 10^\circ \text{ latitude}$. Since we are assuming “first baroclinic mode” structure, we also expect to see twin anticyclones at these locations in the upper troposphere.

The latitudinally averaged streamfunction in the $x-z$ plane, shown in the bottom frame of Fig. 18, shows the twin circulation cells in this plane. The more extensive and (slightly) stronger cell is located to the east, and appears to correspond to the Walker circulation in the atmosphere. It extends a distance of order $7L_E \approx 6000 \text{ km}$, about enough to straddle the equatorial Pacific Ocean. Note, however, that this depends entirely on the value chosen for $\varepsilon$, the dissipation rate. Gill used $\varepsilon = 0.1\sqrt{2}$ (in our notation), so the predicted $e$-folding distance is just $L_E/\varepsilon = 6080 \text{ km}$. Given our time scale of 14 hr, Gill’s value for $\varepsilon$ corresponds to a dissipation time, for both wind and temperature, of 99 hr, rather a short time for both.

### 5.3.2 What does the Gill model mean?

The Gill model is remarkably successful at reproducing major characteristics of the tropical flow, yet it is built on a demonstrably false premise, that one can treat the dynamics a priori as a first baroclinic mode structure. As for convectively coupled waves, it seems better to think of the vertical structure as a consequence of the quasi-equilibrium nature of the tropical atmosphere, where the vertical stratification is constrained by deep moist convection, in which case “1st baroclinic mode-like” dynamical structures are imposed
by the vertical structure of the convection, rather than by real modal constraints.

Note that the zonal length scale of the solution is determined by dissipation, so the fact that the circulation stretches just across the Pacific is by design, in the choice of this parameter. Evidence such as the zonal extent of temperature anomalies during warm El Nino events suggest the real length scale should be larger, and that the zonal extent of the Walker circulation may be limited by convection over tropical South America.

5.4 Monsoon circulations

Fig. 6 shows mean winds in the upper and lower troposphere in northern

![Figure 6: Tropical winds at (top) 200 hPa and (bottom) 850 hPa in JJA.](image)

summer. The most remarkable feature in the circulation over the Indian
Ocean region, with low-level cross-equatorial flow off the east coast of Africa, low-level westerlies along much of the S Asia coast, and a large divergent anticyclone over S Asia and the Arabian peninsula. This is the circulation associated with the Asian summer monsoon, depicted schematically in Fig. 5.4. The upper and low-level flows are connected by deep convection associated with the monsoon over southern Asia and nearby ocean. The Indian Ocean circulation is not the only monsoon circulation on the planet – there are others over West Africa and Central/North America in northern summer, and over northern Australia and the Amazon region in southern summer, but it is by far the most intense.

Understanding the monsoon circulation requires understanding of where the rainfall occurs, since (unlike convection over the warm western Pacific Ocean) it is not a priori obvious why it is located where it is. But a partial understanding can be gotten from a starting point of taking the precipitation as given, and asking what circulation such a pattern of rainfall would drive. To do so, we can turn again to the Gill model.
5.4.1 Gill model solutions

First, let’s consider a forcing antisymmetric about the equator, such that $Q(x, y)$ has no projection onto the even Hermite functions, then the response is qualitatively different. Gill’s solution for the response to such forcing is shown in Fig. 7. (The imposed distribution of $Q$ is approximately indicated by the distribution in the top frame.) Since there is no projection of $Q$ onto $V_0(y)$, there is no Kelvin wave component to the solution and, since this component is the only one with eastward group velocity, there can be no response to the east of the forcing. Near and to the west of the forcing,

Figure 7: Gill’s solution to forcing localized in $x$ and antisymmetric about $y = 0$. 
the response (dominated by the \( n = 2 \) Rossby wave) takes the form of a cyclone/anticyclone pair straddling the equator near \( y = \pm 2 \approx \pm 3200 \) km \( \approx \pm 29^\circ \) latitude, with cross-equatorial flow around and out of the anticyclone and around and into the cyclone.

Now consider an off-equatorial forcing, such as might be associated with monsoonal precipitation. Fig. 8 shows the response to forcing centered in the northern subtropics — in fact, it is simply the sum of the symmetric case of Fig. 5 and the antisymmetric case of Fig. 7. (Again, \( Q \) is approximately shown by the distribution of \( w \).) Recall that this figure, like the others from the Gill model, shows the correct signs for the low-level flow: the upper level flow has the opposite phase. The response shows many of the features characteristic of the Indian Ocean region in northern summer: low level cyclone/upper level anticyclone slightly NW of the upwelling, and cross-

Figure 8: Gill’s solution for response to forcing centered in the northern subtropics.
equatorial flow to the west of the forcing, becoming a southwesterly inflow into the region of low-level convergence. The convergent, low-level cyclone in the region of the heating implies a divergent upper level anticyclone there, as we observe.

A more complete calculation was performed for the globe by Hoskins and Rodwell, illustrated in Fig. 9. Note the extension of the upper level anticyclone over northern Africa, which they suggest contributes to that region’s aridity.
Figure 9: From Hoskins & Rodwell (J. Atmos. Sci., 1995).